Assignment 7

Stars and Universe
January 3, 2025
23-119 ONE CHOI

Problem 1

A star identical to the Sun is in a binary system with a black hole of mass M_H. Assume for simplicity that the density of the star is uniform and that the orbit is circular.

(a) Use Equation 3-9 to obtain the minimum separation from the black hole the star must have in order not to be torn apart by tidal forces.
(b) At what black-hole mass is this minimum separation less than the Schwarzschild radius? Black holes more massive than this can swallow a star whole!

Equation 3-9 states:

$$d = 2.44(\rho_M/\rho_m)^{1/3}R$$

where ρ_M is the average density (SI units = kg/m^3) of the primary and ρ_m is the average density of the satellite.

Solution

The equation envolves in the density of the star and the blackhole, in the case of our problem. Firstly, the density can be determined by using the mass information. For a black hole, the density depends on its mass and Schwarzschild radius:

$$\rho_M = \frac{M_H}{\frac{4}{3}\pi R_S^3},$$

where $R_S = \dfrac{2GM_H}{c^2}$ is the Schwarzschild radius. Substituting this:

$$\rho_M = \frac{M_H}{\frac{4}{3}\pi\left(\frac{2GM_H}{c^2}\right)^3} = \frac{3c^6}{32\pi G^3 M_H^2}.$$

The density of the Sun (ρ_m) is:

$$\rho_m = \frac{M_\odot}{\frac{4}{3}\pi R_\odot^3} = \frac{3M_\odot}{4\pi R_\odot^3}.$$

The tidal disruption distance is then:

$$d = 2.44 \left(\frac{\frac{3c^6}{32\pi G^3 M_H^2}}{\frac{3M_\odot}{4\pi R_\odot^3}} \right)^{1/3} R_\odot = 2.44 \left(\frac{\frac{c^6}{8G^3 M_H^2}}{\frac{M_\odot}{R_\odot^3}} \right)^{1/3} R_\odot.$$

Simplifying,

$$d = 2.44 \left(\frac{c^6 R_\odot^3}{8G^3 M_H^2 M_\odot} \right)^{1/3} R_\odot.$$

(b) The Schwarzschild radius is given as:

$$R_S = \frac{2GM_H}{c^2}.$$

Set $d \leq R_S$ and solve for M_H. From part (a):

$$2.44 \left(\frac{c^6 R_\odot^3}{8G^3 M_H^2 M_\odot} \right)^{1/3} R_\odot \leq \frac{2GM_H}{c^2}.$$

$$\frac{c^6 R_\odot^3}{8G^3 M_H^2 M_\odot} \leq \left(\frac{2GM_H}{2.44 c^2 R_\odot} \right)^3$$

Simplifying in respect to the mass of the blackhole,

$$M_H^5 \geq \frac{c^6 R_\odot^3}{8G^3 M_\odot} \frac{1.22^3 c^6 R_\odot^3}{G^3}$$

$$M_H^5 \geq \frac{1.22^3 c^{12} R_\odot^6}{8G^6 M_\odot}$$

Therefore,

$$M_H \geq 0.7434 \left(\frac{c^{12} R_\odot^6}{G^6 M_\odot} \right)^{1/5}$$

The constants are related in a complex manner, thus it is necessary if the unit matches our expectation of the mass which is kg. Firstly, the unit of the gravitational constant G is:

$$\text{unit}(G) = \frac{\mathrm{N} \cdot \mathrm{M}^2}{\mathrm{kg}^2} = \frac{\mathrm{kg} \cdot \mathrm{m/s} \cdot \mathrm{m}^2}{\mathrm{kg}^2} = \frac{\mathrm{m}^3}{\mathrm{kg} \cdot \mathrm{s}^2}$$

For the fraction in the equation, the unit is expected to be the fifth order of kg. Let's check if the expectation fits well.

$$\text{unit} \left(\frac{c^{12} R_\odot^6}{G^6 M_\odot} \right) = \frac{\mathrm{m}^{12}/\mathrm{s}^{12} \cdot \mathrm{m}^6}{\mathrm{m}^{18}/\mathrm{kg}^6 \mathrm{s}^{12} \cdot \mathrm{kg}} = \mathrm{kg}^5$$

The expactations fits well, we can finally conclude that the values were derived accurately.

Problem 2

(1) The white dwarf Sirius B has a central temperature of $T_c = 3 \times 10^7$ K, a central density of $\rho_c = 6\bar{\rho}$, a radius of $R = 5.5 \times 10^8$ cm, and a mass of $M = 1.1M_\odot$. Here, $\bar{\rho}$ is the average density of the white dwarf. Calculate the gas pressure and radiation pressure at the center of Sirius B, assuming it behaves as an ideal gas. Assume that the white dwarf is composed of carbon, and use an average molecular weight per particle of $\mu = 12/7$. (Hint: $P_{\text{gas}} = nk_BT_c = \frac{\rho_c}{\mu m_H}k_BT_c$, $P_{\text{rad}} = \frac{1}{3}aT_c^4$)

(2) The electron degeneracy pressure is given by:

$$P_{\text{deg}} = 0.0485\frac{h^2}{m_e}\left(\frac{\rho}{\mu_e m_H}\right)^{5/3} \quad \text{(nonrelativistic)}$$

Here, ρ is the density, and μ_e is the average molecular weight per electron. Calculate the nonrelativistic degeneracy pressure at the center of Sirius B. Compare this with the pressures obtained in part (1). Assume that the average molecular weight per electron is $\mu_e = 2$.

Solution

(1) Firstly, the pressure by the gas can be calculated using the ideal gas state equation:

$$P_{\text{gas}} = nk_BT_c = \frac{\rho_c}{\mu m_H}k_BT_c.$$

The density at the center can be calculated as:

$$\rho_c = 6\bar{\rho} = 6 \cdot \frac{M}{\frac{4}{3}\pi R^3} = \frac{9M}{2\pi R^3}$$

Substituting this equation and value into the ideal gas state equation:

$$\therefore P_{\text{gas}} = \frac{9Mk_BT_c}{2\pi\mu m_H R^3} \approx 2.725 \times 10^{21} \text{ Pa}$$

For the pressure by radiation,

$$P_{\text{rad}} = \frac{1}{3}aT_c^4,$$

where a is the radiation density constant and is defined as:

$$a = \frac{4\sigma}{c} = 7.565 \times 10^{-16} \text{ J m}^{-3} \text{ K}^{-4}$$

Directly substituting the value results in:

$$\therefore P_{\text{rad}} = 2.043 \times 10^{14} \text{ Pa}$$

(2) Degeneracy pressure is the pressure that occurs due to the state of the electrons. If the degeneracy pressure get dominated by the thermal pressure (which includes the pressure

due to the gas and the pressure due to the radiation), the star cannot resist the outer pressure, leading to the collapse of the core.

We can calculate the degeneracy pressure and compare the value with the pressure we have calculated above. Degeneracy pressure is expressed as:

$$P_{\text{deg}} = 0.0485 \frac{h^2}{m_e} \left(\frac{\rho_c}{\mu_e m_H} \right)^{5/3} = 0.0485 \frac{h^2}{m_e} \left(\frac{9M}{2\pi R^3 \mu_e m_H} \right).$$

Substitute: $h = 6.626 \times 10^{-34}$ J·s, $m_e = 9.11 \times 10^{-31}$ kg, $\rho_c = 1.884 \times 10^{10}$ kg/m^3, $\mu_e = 2$, $m_H = 1.67 \times 10^{-27}$ kg.

$$\therefore P_{\text{deg}} = 6.257 \times 10^{25} \text{ Pa}$$

To compare the value of the pressure, it can be summarized as below:

Table 1. Summary of Pressure at the Center of Sirius B

Pressure Type	**Value (Pa)**
Gas Pressure (P_{gas})	2.725×10^{21}
Radiation Pressure (P_{rad})	2.043×10^{14}
Degeneracy Pressure (P_{deg})	6.257×10^{25}

According to what I've calculated, the value of degeneracy pressure is higher than the gas pressure plus the radiation pressure. This means that the star is stablely sustaining its pressure.

Assignment 1

Stars and Universe
April 16, 2025
23-119 ONE CHOI

Problem 1

Using the given data, plot the orbit of Mercury (using a computer program). Find the semi-major axis, semi- minor axis, and eccentricity of Mercury from the given data, compare them with the actual values, and describe the cause of any error. Assume that the Earth moves in a uniform circular motion around the Sun at a distance of 1 AU.

Table 1. Julian Date and Maximum Elongation of Mercury

Date	Julian Date	Max. Elongation
1979.03.08	2443941	18° E
1979.04.21	2443985	27° W
1979.07.03	2444058	26° E
1979.08.19	2444105	19° W
1979.10.29	2444176	24° E
1979.12.07	2444215	21° W
1980.02.19	2444289	18° E
1980.04.02	2444332	28° W
1980.06.14	2444405	24° E
1980.08.01	2444453	19° W
1980.10.11	2444524	25° E
1980.11.19	2444563	20° W

Solution

In the late 16th century, when Tycho Brahe studied astronomy, the only thing to believe and observe stars were their eyes; no telescopes or DSLR cameras. Tycho Brahe had left tons of observation records and handed them over to his student, Johannes Kepler utilized mathematical techniques to analyze Tycho Brahe's observation record and find out elementary astronomical laws.

The records above are examples of Mercury's elongation (maximum) and its date. Using this, we can extract information about the location of Mercury. Let Earth be expressed as E, Sun be expressed as S, and Mercury be expressed as M in the figure below. Using geometry, we can find out the cartesian coordinate of Mercury on each date and analyze its orbit.

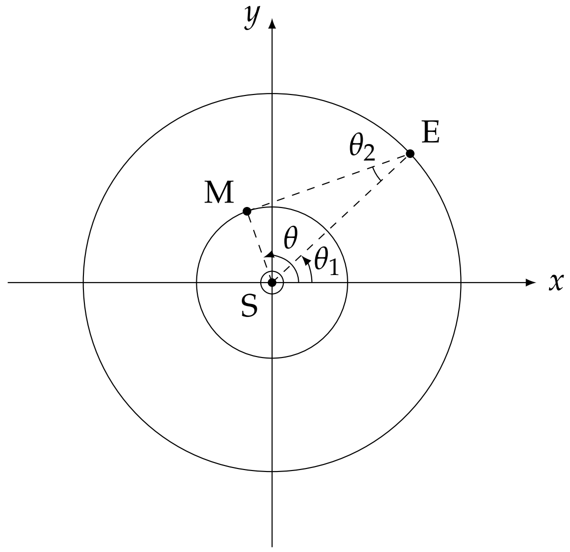


Figure 1: 2-Dimensional Diagram of Sun, Earth, and Mercury

Let the angle of Mercury from x-axis be *theta* and the angle of the Earth from x-axis be $theta_1$. Angle θ_1 will be calculated using the Julian date soon. If we are able to express the angle θ and the distance R from the Sun using θ_1, then, the cartesian coordinate of Mercury is $(R\sin\theta, R\cos\theta)$.

Since the angle of Mercury observed from the Earth is the maximum elongation, $\angle$SME should be 90°. Therefore,

$$\theta = \theta_1 + \left(\frac{\pi}{2} - \theta_2\right)$$

θ_2 can be extracted from the **Table 1**. Since counter-clockwise is positive sign for angle in standard, eastern elongation will be a positive angle, and the western elongation will be a negative angle.

θ_1 on the other hand can be calculated using Julian date. Precisely, 1 year is equivalent to 365.242190 days, and let's consider the "0th" Julian day to be at angle 0. Then, the angle θ can be written by:

$$\theta = \frac{\text{Julian day}}{365.242190}.$$

R, in other words $\overline{\text{SM}}$, can be simply calculated using trigonometric relation. Since $\triangle$SME is a right triangle, $R\ (AU) = 1\ (\text{AU}) \times \sin\theta_2$.

To write all organized raw data into table, it looks like following (**Table 2**):

Table 2. Raw Data for Mercury

Julian	θ	θ_1	θ_2	R	M_x	M_y
2443941	42044	42043	0.314	0.309	-0.308	0.0216
2443985	42045	42043	-0.471	0.454	-0.0446	-0.452
2444058	42045.7	42044.6	0.454	0.438	0.101	-0.427
2444105	42047.3	42045.4	-0.332	0.326	0.315	0.0823
2444176	42047.8	42046.7	0.419	0.407	0.304	0.270
2444215	42049.3	42047.3	-0.367	0.358	-0.206	0.293
2444289	42049.9	42048.6	0.314	0.309	-0.288	0.111
2444332	42051.4	42049.3	-0.489	0.469	-0.181	-0.433
2444405	42051.7	42050.6	0.419	0.407	-0.0189	-0.406
2444453	42053.3	42051.4	-0.332	0.326	0.325	-0.0134
2444524	42053.8	42052.6	0.436	0.423	0.387	0.169
2444563	42055.2	42053.3	-0.349	0.342	-0.100	0.327

M_x, M_y: x and y coordinate of Mercury's location respectively

We need to fit these observatory data points into the ellipse. Various methods exist to fit multiple data points, but each of the methods has its advantages and disadvantages. For problem 1, there are around 10 data points, meaning that each of the data fitting will not have a huge impact on the overall ellipse, yet in problem 2, which will be mentioned soon, it only has 8 points with 3 different positions of Mars. For 3 data points, ellipse fitting can lead to incorrect fitting.

So I have used a method of using the fact that the location of the Sun is one of the foci on the ellipse. Let the Sun be located at $(0,0)$. and the unit for distance is AU (Astronomical Unit). To use the fact that the $(0,0)$ is one of the foci on the ellipse, the easiest way is to use the polar equation. The equation for ellipse in the polar coordinate is:

$$r = \frac{ed}{1 + e\cos\theta}, \quad \text{or} \quad r = \frac{a(1-e^2)}{1 + e\cos\theta}$$

To draw this on the plane, it looks like following:

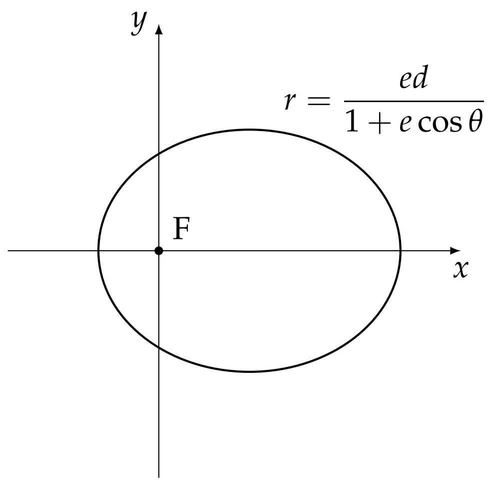


Figure 2: Ellipse in Polar Form

The advantage of the polar equation of the ellipse is that one of the foci is always locked onto $(0,0)$, which is the situation that is given in the question. On the other hand, cartesian coordinate requires a shift to either right or left in order to make a foci be on the origin.

The mechanism of ellipse fitting is mostly done by the computer and the code, yet here is a simple outline of how curve fitting is done.

Firstly, data points are converted into np.array, which is used as an x and y input for Python analysis. Secondly, since the coordinates are based on a cartesian basis, we have to convert them into polar form. The conversion formula for it is:

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \arctan\left(\frac{y}{x}\right)$$

The generalize form for ellipse is:

$$r_{\text{model}} = \frac{a(1-e^2)}{1+e\cos\theta}$$

We let this equation be r_{model}, the value that data points be approaching. Residual is value that leftover from r_{model}, allowing both positive and negative values. If we sum it all up,

$$\text{Residual} = \sum \left(r_i - \frac{a(1-e^2)}{1+e\cos(\theta+\phi)} \right).$$

Here, ϕ is introduced as a parameter that explains how much ellipse has been rotated. Then, we need to set up initial guesses for a (semi-major axis), e (eccentricity), and ϕ (rotation angle) so that the code reduces the residual starting from this value. It is important not to set the value too extreme. The initial values that I have used are:

$$a_{\text{initial}} = 0, \quad e_{\text{initial}} = 0.5, \quad \text{and} \quad \phi_{\text{initial}} = 0.$$

Finally, the code uses its algorithm to reduce the residual as much as possible. After it is done, I made it so that it converts into cartesian coordinates again, and prints out essential information about the ellispe.

To provide the full code, it looks like following:

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.optimize import minimize

xm = np.array([-0.308261386, -0.044610083, 0.100945085, 0.314999963, \
               0.304064775, -0.205857618, -0.288487754, -0.180563018, \
               -0.01892744, 0.325292916, 0.387198293, -0.100368388])
ym = np.array([0.021596771, -0.451793442, -0.42659038, 0.082277862, \
               0.270146829, 0.293343192, 0.110753412, -0.433359602, \
               -0.406296011, -0.013384383, 0.169362562, 0.326961718])

xM = np.array([0.583387376, 0.589163815, 0.582676624, 0.534235559, \
               0.534138349, 0.535807774, -1.649739552])
yM = np.array([-1.282201815, -1.286879484, -1.284389284, 1.434685304, \
               1.44025444, 1.435667733, 0.167532682])

x, y = xm, ym

r = np.sqrt(x**2 + y**2)
```

```python
theta = np.arctan2(y,x)

#Fitting
def fitting(params, angle, r):
    a, e, phi = params
    residuals = []
    for i, j in zip(angle,r):
        rotate = i-phi
        polar = a*(1-e**2)/(1+e*np.cos(rotate))
        residuals.append((j-polar)**2)
    return np.sum(residuals)

guess = [np.mean(r),0.5,0] # a,e,phi initial
bound = [(0,None), (0,1), (None, None)]
result = minimize(fitting, guess, args=(theta,r), method='L-BFGS-B', bounds=bound)
a_opt, e_opt, phi_opt = result.x

# Ellipse
thetaEllipse = np.linspace(0, 2*np.pi, 500)
rEllipse = a_opt*(1-e_opt**2)/(1+e_opt*np.cos(thetaEllipse))

xEllipse = rEllipse*np.cos(thetaEllipse)
yEllipse = rEllipse*np.sin(thetaEllipse)

xRotated = xEllipse*np.cos(phi_opt) - yEllipse*np.sin(phi_opt)
yRotated = xEllipse*np.sin(phi_opt) + yEllipse*np.cos(phi_opt)

# Earth
thetaCircle = np.linspace(0, 2*np.pi, 500)
xCircle = np.cos(thetaCircle)
yCircle = np.sin(thetaCircle)

# Plot
plt.figure(figsize=(8, 8))
plt.plot(x, y, 'ro', label='Data Points')
plt.plot(xRotated, yRotated, 'b-', label='Fitted Ellipse')
plt.plot(xCircle, yCircle, 'g-', label='Earth')
plt.scatter(0, 0, color='yellow', label='Sun (0,0)', s=100)
plt.title('Mercury Orbit Fitting', pad=20, loc='center')
plt.xlabel('X (AU)')
plt.ylabel('Y (AU)')
plt.gca().set_aspect('equal', adjustable='box')
plt.legend()
plt.legend(bbox_to_anchor=(1.05,1.0), loc='upper left')
plt.tight_layout()
plt.grid(True)

plt.savefig("Mercury.svg", format="svg")
plt.show()

b_opt = a_opt * np.sqrt(1 - e_opt**2)

print(f"e = {e_opt}, phi = {phi_opt}")
print(f"a = {a_opt}")
print(f"b = {b_opt}")
```

After running the full code, the results are saved as SVG file. It shows the fitted ellipse and the Earth's orbit for comparison. Details of the elliptical orbit is provided as printed value in python. Here is the result after running the code.

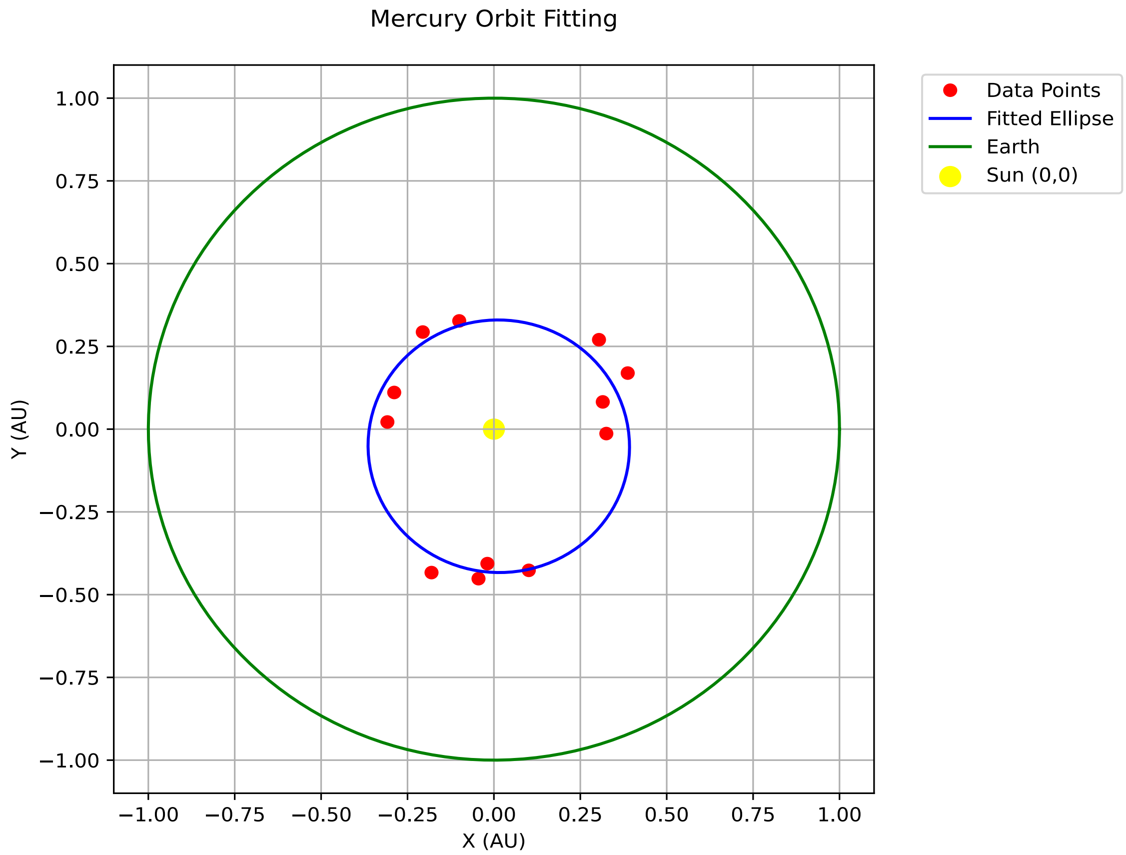


Figure 3: Mercury Orbit Fitting

The green orbit is the Earth's orbit and the blue orbit is the Mercury's orbit. And it appears to fit the red data points accurately. To provide the information about this ellipse (experimental) and currently accepted values for Mercury's orbit (theoretical):

Table 3. Mercury Orbit's Physical Constant

Constants	Experimental	Theoretical
Eccentricity (e)	0.14038	0.20563
Semi-major axis (a)	0.38183 AU	0.38710 AU
Semi-minor axis (b)	0.37805 AU	0.37883 AU

The calculation seems to be done precisely and correctly. Even though the value of eccentricity shows quite a difference, comparing the semi-major axis and semi-minor axis, each of the experimental values is extremely close to the theoretical value.

The error seems to be caused by several places. Since the calculation for the position is pure geometry and mathematics, no error has been involved during this process. The main error seems to be caused by the **inaccruate observation**. The significant figure for the angle is lacking; 2. An accurate measurement of the angle with a more significant figure would have led to reduced error and decreased variance within the data points.

There are lots of minor error-causing factors, on the other hand. Firstly, even though the sun is massive ($M_\odot = 1.989 \times 10^{30}$ kg), due to its binary mass system orbit, the sun also moves in an elliptical path. Secondly, even though the observation itself may be accurate, the atmosphere of the Earth leads to atmospheric refraction. The formula for refraction is:

$$R = \cot\left(h_a + \frac{7.31}{h_a + 4.4}\right)$$

where R is the refraction angle in radians and h_a is the astronomical body's altitude. We can calibrate the actual angle of Sun - Earth - Mercury using this formula. In addition to this, Earth's revolution around the sun during the day can lead to slight angle distortion. While the value itself is limited to $0^\circ < \theta_{\text{distorted}} < 1^\circ$, the distance from two planets is so distant that a slight viewing angle leads to a huge tangential distance. Besides, there are countless minor error factors that affect this problem's result.

Problem 2

Plot the orbit of Mars using the given data. Calculate the semi-major axis, semi-minor axis, and eccentricity of Mars from the given data, compare them with the actual values, and explain the causes of any error. Assume that the Earth moves in a uniform circular motion around the Sun at a distance of 1 AU.

Table 4. Julian Date and Elongations for Mars

Date	Julian Date	S-E-M Elongation	Position of E from S
1969.08.17	2440451	108°	324°(-36°)
1971.07.05	2441138	219°(-141°)	282°(-78°)
1973.05.22	2441825	278°(-82°)	241°(-119°)
1974.01.02	2442050	111°	101°
1975.11.20	2442737	214°(-146°)	57°
1977.10.07	2443424	276°(-84°)	14°
1980.04.09	2444339	-234°(126°)	199°(-161°)
1982.02.25	2445026	-138°	156°

S: Sun, E: Earth, M: Mars

Solution

Let's take into account that the period for Mars's revolution around the Sun is 687 days. The reason why we need this information is because we don't have extra information about Mars like we did on Mercury, which they were always on the maximum elongation. Here, I will use two different observation data to predict one Mars position. But the condition is that Mars should finish (at least) one revolution. Then, I will analyze using geometry to find Mars's location.

In addition, if we know one relative revolution from the Earth, by using the synodic period equation, the period of Mars's revolution around the sun can be determined. The formula is presented as:

$$\frac{1}{S} = \left|\frac{1}{P} - \frac{1}{E}\right|$$

where S, P, E is a synodic period of the Sun, a planet, and the Earth respectively. The sign is determined by whether the planet is an inferior planet or a superior planet. Now, let's see the number of days between each observation.

Table 5. Julian Date and Date Difference for Each Observation

Julian Date	Date Difference
2440451	0
2441138	687
2441825	687
2442050	225
2442737	687
2443424	687
2444339	915
2445026	687

Here, 1st, 2nd, and 3rd observation is recorded exactly after 687 days, meaning that they are located at the same position. However, the 3rd and 4th observation is recorded after 225 days, and the 6th and 7th observation is recorded after 915 days, meaning that there will be largely 3 different positions on the graph. Even though there are 3 main points on the graph, we can extract ${}_{n}C_{2}$ data points per main point, because we can choose 2 of 3 points to determine Mars's coordinate.

Refer to the figure down below. If the Earth is originally located at point E and then moved to E′, we can draw a quadrangle. Let θ be the elongation of Earth - Sun - Earth′, ϕ_1 and ϕ_2 be the elongation of Sun - Earth - Mars. Use the fact that the distance between the Sun and the Earth is 1 AU, in other words, $\overline{\text{SE}} = 1$.

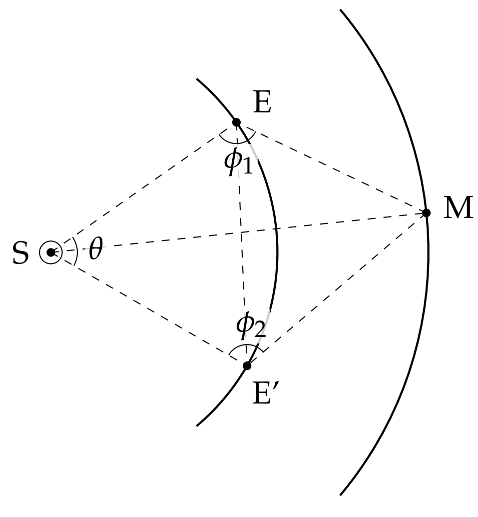


Figure 4: Sun, Earth, Mars and its Geometric Relation 1

$\triangle$ESE′ is an isosceles triangle, therefore,

$$\angle \text{SEE}' = \angle \text{SE}'\text{E} = \frac{\pi - \theta}{2}$$

Using cosine law, if $\overline{\text{EE}'} = L$:

$$L = \sqrt{1^2 + 1^2 - 2 \cdot 1 \cdot \cos\theta} = \sqrt{2 - 2\cos\theta}$$

Let $\angle$MEE′ $= \alpha$ and $\angle$ME′E $= \beta$. Then,

$$\alpha = \phi_1 - \frac{\pi - \theta}{2} \quad \text{and} \quad \beta = \phi_2 - \frac{\pi - \theta}{2}.$$

Then, we are able to solve $\triangle$MEE′. After solving it, we can solve for $\triangle$SEM to obtain the length $\overline{\text{SM}}$, which equals to R.

Let $\overline{\text{ME}} = a$ and $\overline{\text{ME}'} = b$. By examining $\triangle$MEE′, we get:

$$a \cos\alpha + b\cos\beta = L$$

$$a \sin\alpha = b\sin\beta$$

Use the method of eliminating b,

$$b\sin\beta \cdot \cot\beta = b\cos\beta = a\sin\alpha\cot\beta$$

Eliminate b and express L by a.

$$L = a\cos\alpha + a\sin\alpha\cot\beta$$

Express a using L

$$\begin{aligned} a &= \frac{L}{\cos\alpha + \sin\alpha\cot\beta} \\ &= \frac{L\sin\beta}{\sin\beta\cos\alpha + \sin\alpha\cos\beta} \\ &= \frac{L\sin\beta}{\sin(\alpha+\beta)} \end{aligned}$$

Using cosine law in $\triangle$SEM, we can determine R. Use that $L = \sqrt{2 - 2\cos\theta}$ and $\alpha + \beta = \phi_1 + \phi_2 - \pi + \theta$.

$$\begin{aligned} R^2 &= 1^2 + a^2 - 2a\cos\phi_1 \\ &= 1 + \frac{L^2\sin^2\beta}{\sin^2(\alpha+\beta)} - \frac{2L\sin\beta\cos\phi_1}{\sin(\alpha+\beta)} \\ &= 1 + \frac{2(1-\cos\theta)\sin^2\left(\phi_2 - \frac{\pi-\theta}{2}\right)}{\sin^2(\phi_1+\phi_2-\pi+\theta)} - \frac{2\sqrt{2(1-\cos\theta)}\sin\left(\phi_2 - \frac{\pi-\theta}{2}\right)\cos\phi_1}{\sin^2(\phi_1+\phi_2-\pi+\theta)} \end{aligned}$$

Even though it looks wrong or able to be simplified, no error had occurred during the calculation.

$$\therefore R = \sqrt{1 + \frac{2(1-\cos\theta)\sin^2\left(\phi_2 - \frac{\pi-\theta}{2}\right)}{\sin^2(\phi_1+\phi_2-\pi+\theta)} - \frac{2\sqrt{2(1-\cos\theta)}\sin\left(\phi_2 - \frac{\pi-\theta}{2}\right)\cos\phi_1}{\sin^2(\phi_1+\phi_2-\pi+\theta)}}$$

Since we have determined R in terms of the measurement, we have to calculate x and y coordinate of the Mars. If the angle of x-axis, the Sun, and the Mars is θ_M, x and y coordinates would be:

$$(R\cos\theta_M, R\sin\theta_M)$$

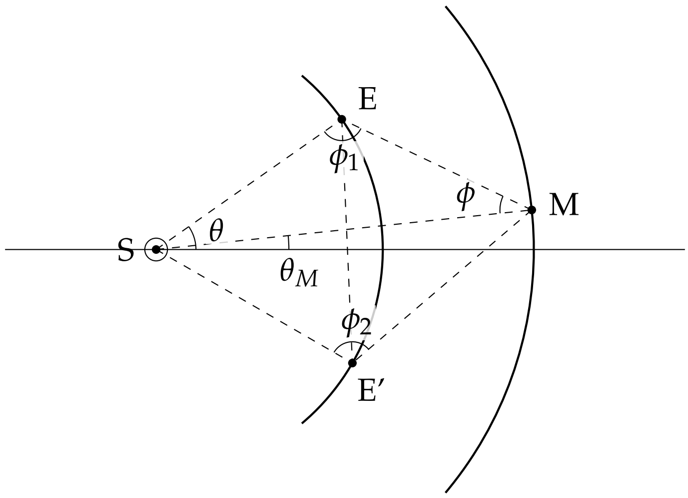


Figure 5: Sun, Earth, Mars and Its Geometric Relation 2

Refer to the figure above. According to this,

$$\theta_M = \theta + \phi_1 + \phi - \pi$$

Since θ, ϕ_1 are all observations, we only need to calculate ϕ. Use sine's law on $\triangle$SEM to determine this angle.

$$\frac{\sin\phi_1}{R} = \frac{\sin\phi}{1}$$

Therefore,

$$\phi = \arcsin\left(\frac{1}{R}\sin\phi_1\right)$$

Hence, the cartesian coordinate for Mars is:

$$(M_x, M_y) = \left(R\cos\left(\theta + \phi_1 + \arcsin\left(\frac{1}{R}\sin\phi_1\right) - \pi\right), R\sin\left(\theta + \phi_1 + \arcsin\left(\frac{1}{R}\sin\phi_1\right) - \pi\right)\right)$$

After all these horrendous calculations, we have to convert it into Excel for data processing. Things to be cautious of here is that we should input $2\pi - \phi_2$, since ϕ_2 also represents a positive angle clockwise. For easier data viewing, I will name 3 main positions P_1, P_2, P_3 respectively. Firstly, for P_1, these are the raw data after calculation.

Table 6. Angles and Distance to Mars for P_1

Observation No.	θ	ϕ_1	ϕ_2	R
1st & 2nd	0.73304	1.88496	2.46091	1.40868
2nd & 3rd	0.71558	3.82227	1.43117	1.41533
1st & 3rd	1.44862	1.88496	1.43117	1.41038

$\sigma = 0.003457$

Secondly, for P_2,

Table 7. Angles and Distance to Mars for P_2

Observation No.	θ	ϕ_1	ϕ_2	R
4th & 5th	0.76795	1.93732	2.54818	1.53092
5th & 6th	0.75049	3.73501	1.46608	1.53611
4th & 6th	1.51844	1.93732	1.46608	1.53239

$\sigma = 0.002673$

And finally for P_3,

Table 8. Angles and Distance to Mars for P_3

Observation No.	θ	ϕ_1	ϕ_2	R
7th & 8th	0.75049	-4.08407	8.69174	1.65822

$\sigma = \mathrm{N/A}$

Then, we are able to get all the values and convert it into the cartesian coordinates. This process was also done by Microsoft Excel.

Table 9. Cartesian Coordinates for Mars

Data No.	θ_M	M_x	M_y
1st	5.13938	0.58339	-1.28220
2nd	5.14173	0.58916	-1.28688
3rd	5.13828	0.58268	-1.28439
4th	1.21433	0.53424	1.43469
5th	1.21566	0.53414	1.44025
6th	1.21360	0.53581	1.43567
7th	-3.24280	-1.64974	0.16753

I have used the same code for the curve fitting. Mars also has the Sun as one of the foci of the orbit, thus it's applicable. Instead, I had to change

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.optimize import minimize

xm = np.array([-0.308261386, -0.044610083, 0.100945085, 0.314999963, \
             0.304064775, -0.205857618, -0.288487754, -0.180563018, \
             -0.01892744, 0.325292916, 0.387198293, -0.100368388])
ym = np.array([0.021596771, -0.451793442, -0.42659038, 0.082277862, \
             0.270146829, 0.293343192, 0.110753412, -0.433359602, \
             -0.406296011, -0.013384383, 0.169362562, 0.326961718])

xM = np.array([0.583387376, 0.589163815, 0.582676624, 0.534235559, \
             0.534138349, 0.535807774, -1.649739552])
yM = np.array([-1.282201815, -1.286879484, -1.284389284, 1.434685304, \
             1.44025444, 1.435667733, 0.167532682])

x, y = xM, yM
```

line 17 with the code above to apply Mars's data. The result is saved as SVG file, and it appears like the following figure:

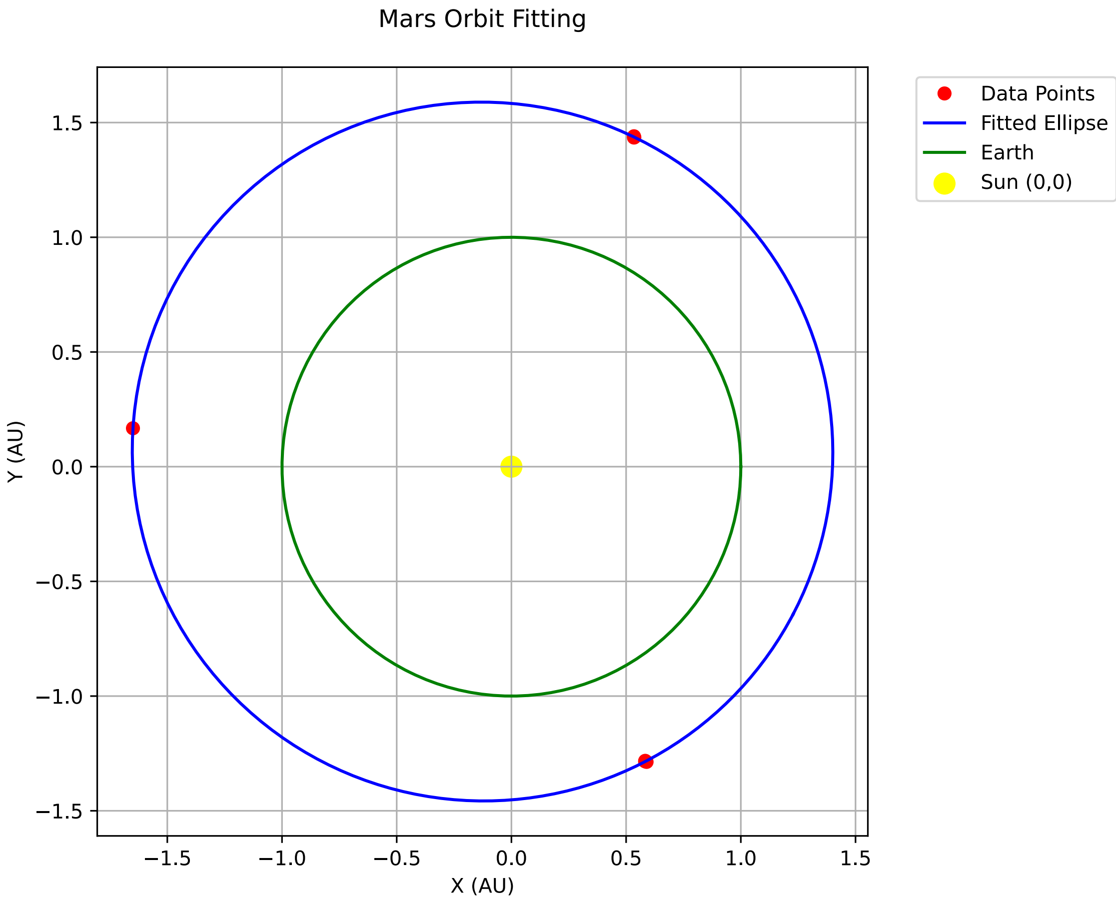


Figure 6: Mercury Orbit Fitting

It also appears to t the red data points accurately. To provide the information about this ellipse (experimental) and currently accepted values for Mars's orbit (theoretical):

Table 10. Mars Orbit's Physical Constant

Constants	Experimental	Theoretical
Eccentricity (e)	0.0930	0.0934
Semi-major axis (a)	1.5283 AU	1.5237 AU
Semi-minor axis (b)	1.5217 AU	1.5170 AU

Mars's data seem to fit the theoretical value extremely accurately. It implies that the calculation and data fitting is done successfully.

Problems 1 and 2 both deal with the same type of question, thus the error-causing factor is the same. To restate it, the major error-causing factor is **inaccurate measurement**, and the minor factors include atmospheric refraction, Earth's revolution during the day, and uncorrectable telescopes, etc. From two problem sections, I have successfully determined the position and information of Mercury's and Mars's orbit and illustrated error-causing factors.

Assignment 2

Stars and Universe
February 21, 2025
23-119 ONE CHOI

Problem 1

Given the initial velocity $\boldsymbol{v_0}$ and initial position $\boldsymbol{r_0}$, find the 6 orbital elements.

Solution

To fully illustrate the orbit in 3-Dimensional space, we need orbital elements, such as coordinates. These elements are called the Classical Orbital Elements, or COE for abbreviation. COE includes: eccentricity (e), semi-major axis (a), inclination (i), longitude of the ascending node (Ω), argument of periapsis (ω), and time of passing perihelion (τ). Using the figure allows better understanding for COE.

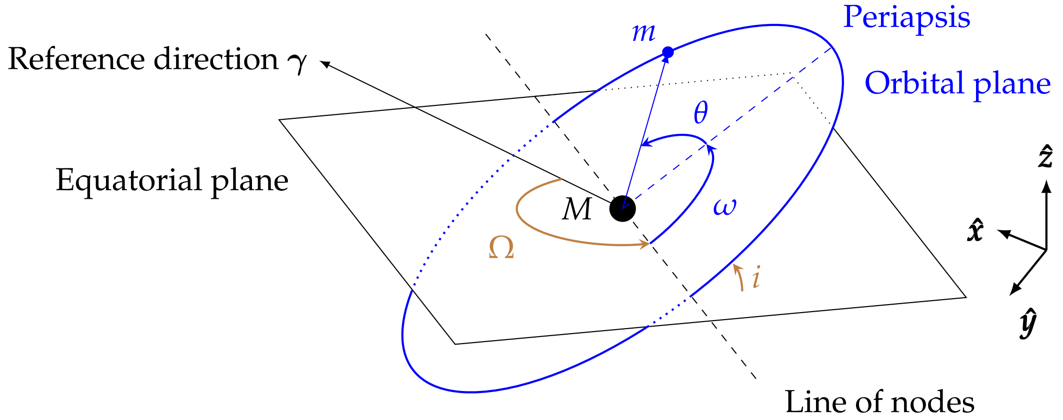


Figure 1: Classical Orbital Elements

Specific Angular Momentum
The reason why 'specific' is playing a role as a prefix is because it is divided by a mass. It is often efficiently used in astrophysics where mass doesn't affect its acceleration of the orbit, usually when it is sufficiently trivial compared to the celestial body's mass.

In the unknown orbit, $\boldsymbol{v_0}$ and $\boldsymbol{r_0}$ is given. The first thing to notice here is that the specific angular momentum vector. In an orbit that is caused by the gravity, angular momentum $\boldsymbol{L}$

of the mass is constant since no external torque is acting on the mass to work.

$$\sum \boldsymbol{\tau} = \frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t}$$

$$\therefore \frac{\mathrm{d}\boldsymbol{h}}{\mathrm{d}t} = \frac{1}{m_2}\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = 0$$

According to the definition of specific angular momentum,

$$\boldsymbol{h} = \boldsymbol{r_0} \times \boldsymbol{v_0}$$

Here, the sign of z coordinate of $\boldsymbol{h}$ determines whether the orbit is clockwise or counter-clockwise, to say it simply.

Inclination
Note that $\boldsymbol{h}$ is a perpendicular vector of orbiting plane. Therefore, utilizing this vector will lead to the equation of inclination. Using geometry, we get:

$$i = \arccos\left(\frac{h_z}{h}\right)$$

Ascending Node
Ascending node is an angle between the x-axis, or the reference direction. If we created a vector that points to line of node, Ω would be an angle between the x-axis and that vector. Let a unit vector that points to line of node be $\hat{\boldsymbol{n}}$. This unit vector should be the line where periapsis begins. Using $\boldsymbol{h}$, $\hat{\boldsymbol{n}}$ can be defined.

$$\hat{\boldsymbol{n}} \equiv \frac{\hat{\boldsymbol{z}} \times \boldsymbol{h}}{|\hat{\boldsymbol{z}} \times \boldsymbol{h}|}$$

Here, Ω is an angle in between x-axis and $\hat{\boldsymbol{n}}$. Thus, following satisfies:

$$\hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{n}} = \cos\Omega$$

Transforming into an equation of Ω,

$$\Omega = \arccos\left(\hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{n}}\right)$$

Yet, the thing to remark is that the range of Ω is $[0, 2\pi]$, while the range of inverse cosine function is $[0, \pi]$. Rest of the angle is determined by the y coordinate of $\hat{\boldsymbol{n}}$. To sum up,

$$\Omega = \begin{cases} \arccos\left(\hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{n}}\right), & \text{if} \quad n_y > 0 \\ 2\pi - \arccos\left(\hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{n}}\right), & \text{if} \quad n_y < 0 \end{cases}$$

This idea is easily comprehensible using figure 1.

Eccentricity
Eccentricity of the orbit can be calculated easily using the generalized form for it. The generalized equation for eccentricity involves in vis-viva equation. This equation will be rigorously proven in problem 2, hence let's accept that vis-viva equation is true. Vis-viva equation states:

$$v^2 = \mu\left(\frac{2}{r} - \frac{1-e^2}{p}\right)$$

Rewriting equation, we claim:

$$e = \sqrt{1 + p\left(v^2 - \frac{2\mu}{r}\right)}$$

Semi-latus rectum can also be expressed as following:

$$p = \frac{h^2}{\mu}$$

This equation will also be proven in problem 2. At this point, eccentricity is all presented using information we know.

Semi-Major Axis
The semi-major axis of ellipse is expressed as:

$$a = \frac{p}{1-e^2} = \frac{h^2}{\mu\left(1-e^2\right)}$$

Hence, it is determined.

Argument of Periapsis
Literally stated, argument of periapsis is the angle between the line of nodes and periapsis. Therefore, we need to introduce a vector that points toward to periapsis for better analysis. The eccentricity is a scalar, yet eccentricity vector can also be defined, where its norm is the eccentricity. Eccentricity vector is defined as following:

$$\boldsymbol{e} = \frac{\boldsymbol{v} \times \boldsymbol{h}}{\mu} - \frac{\boldsymbol{r}}{r}$$

There is an algebraic method of explaining that this indeed points periapsis points and its norm is the eccentricity, but I found it very un-intuitive and lack a rigor. So, proof is excluded. Besides, we know initial velocity, initial position, and specific angular momentum. Substituting this:

$$\boldsymbol{e} = \frac{\boldsymbol{v_0} \times \boldsymbol{h}}{\mu} - \frac{\boldsymbol{r_0}}{r}.$$

Same algorithm works for argument of periapsis. Let

$$\hat{\boldsymbol{e}} = \frac{\boldsymbol{e}}{|\boldsymbol{e}|},$$

then,

$$\boldsymbol{e} \cdot \boldsymbol{n} = \cos\omega$$

Before applying inverse of cosine, ω's range is $[0, 2\pi]$, and the ambiguity of cosine is determined by the direction of the orbit, in other words, the sign of the z coordinate of specific angular momentum. Whether the orbit is clockwise or counter-clockwise if $h_z = 0$ is just a matter of way of defining it, so it is not a huge deal. To sum up,

$$\omega = \begin{cases} \arccos\left(\hat{\boldsymbol{e}} \cdot \hat{\boldsymbol{n}}\right), & \text{if} \quad h_z > 0 \\ 2\pi - \arccos\left(\hat{\boldsymbol{e}} \cdot \hat{\boldsymbol{n}}\right), & \text{if} \quad h_z < 0 \end{cases}$$

True Anomaly
The final step is to determine the argument from periapsis. This process requires a current position vector. Since we don't have live position of the mass, let's calculate a true anomaly for the initial position. Same algorithm, true anomaly is an angle between the eccentricity vector and the position vector. Now considering the range of true anomaly, the ambiguity of cosine is determined by the cross product of the position vector and the eccentricity vector. In conclusion:

$$\theta = \begin{cases} \arccos\left(\hat{\boldsymbol{e}} \cdot \hat{\boldsymbol{r}}\right), & \text{if} \quad (\hat{\boldsymbol{e}} \times \hat{\boldsymbol{r}}) \cdot \hat{\boldsymbol{z}} > 0 \\ 2\pi - \arccos\left(\hat{\boldsymbol{e}} \cdot \hat{\boldsymbol{r}}\right), & \text{if} \quad (\hat{\boldsymbol{e}} \times \hat{\boldsymbol{r}}) \cdot \hat{\boldsymbol{z}} < 0 \end{cases}$$

However, for true anomaly, various exceptions exists, which interrupts quick calculation. The equation written above is a simple equation for θ without considering specific situation, but when situation gets specific, cosine's ambiguity should be determined by using different conditions.

From this, we have successfully derived every 6 orbital elements just by manipulating the initial velocity and position.

Problem 2

Lets look at the relative motion of a celestial body with mass m_2 relative to a celestial body with mass m_1.

(1) Show that the characteristics of the relative orbital motion of a celestial body m_2 appear as follows for each conic, that is, circular, elliptical, form line, and hyperbolic orbit.

Table 1. Characteristics of relative orbital motion

Conic Section	e	p	$\dot{A}$	$\boldsymbol{h}$	T	v^2
Circle	0	a	$\frac{1}{2}\sqrt{\mu a}$	$\sqrt{\mu a}$	$-\frac{\mu m_2}{2a}$	$\frac{\mu}{a}$
Ellipse	$0 < e < 1$	$a(1-e^2)$	$\frac{1}{2}\sqrt{\mu a(1-e^2)}$	$\sqrt{\mu a(1-e^2)}$	$-\frac{\mu m_2}{2a}$	$\mu\left(\frac{2}{r} - \frac{1}{a}\right)$
Parabola	1	$2q$	$\sqrt{\frac{1}{2}\mu q}$	$\sqrt{2\mu q}$	0	$\frac{2\mu}{r}$
Hyperbola	$e > 1$	$a(e^2-1)$	$\frac{1}{2}\sqrt{\mu a(e^2-1)}$	$\sqrt{\mu a(e^2-1)}$	$\frac{\mu m_2}{2a}$	$\mu\left(\frac{2}{r} + \frac{1}{a}\right)$

e: Eccentricity, p: Semi-Latus Rectum, $\dot{A}$: Velocity of Area, $\boldsymbol{h}$: Specific Angular Momentum
T: Total Energy, v^2: Velocity Squared

(2) If the celestial body m_2 moves in an ellipse with respect to the celestial body m_1, apply

the area of the ellipse and Kepler's third law to derive the following:

$$4\dot{A}^2 = \mu p$$

Here, $\dot{A}$ is a velocity of area, p is semi-latus rectum, and $\mu = G(m_1 + m_2)$

Solution

(1) Let's prove their physical characteristics one by one.

Eccentricity

The eccentricity can be determined by the different definition of the conic section. Therefore, eccentricity is trivial to prove.

Semi Latus Rectum, p

The basic equation for the orbital motion is:

$$r = \frac{h^2/\mu}{1 + e\cos\theta}$$

where $h = |\mathrm{h}| = |\mathrm{r} \times \mathrm{v}|$ and $\mu = Gm_1$. Semi latus rectum (p) is defined as a vertical distance between the orbit and a focus. Applying $\theta = \pi/2$ to determine p, we get:

$$p = \frac{h^2}{\mu}$$

Therefore,

$$h = \sqrt{\mu p}$$

For circle, it is obvious that $p = a$. For the rest of the shape, proof is necessary.

Ellipse
Geometric relation can easily determine the relation between semi-major axis and eccentricity.

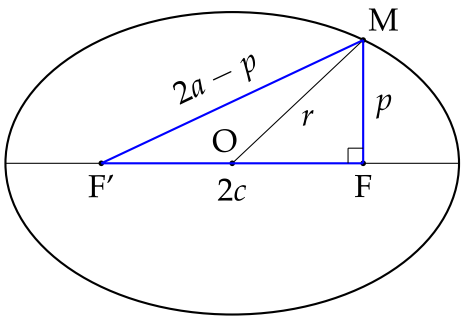


Figure 2: Conic Section

Due to the definiton of ellipse, $\overline{\text{MF}} + \overline{\text{MF}'} = 2a$ (no matter where M is located on the trace of the ellipse). Using a Pythagorean theorem, we obtain:

$$p^2 + 4c^2 = (2a - p)^2$$

The relation between semi-major axis and focal length is:

$$c = \sqrt{a^2 - b^2} = a\sqrt{1 - \frac{b^2}{a^2}} = ae$$

Substituting this into the Pythagorean theorem, we get:

$$p^2 + 4e^2a^2 = 4a^2 - 4ap + p^2, \quad \text{therefore} \quad p = a(1 - e^2).$$

Parabola
A parabola is defined as a set of points, eventually a curve, such that it creates an equal distance from a fixed point (focus) and from a fixed straight line (directrix).

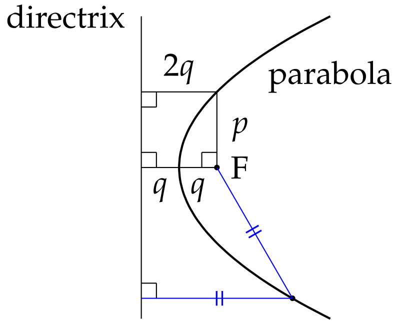


Figure 3: parabola

Using the definition of parabola, we simple get

$$p = 2q$$

Hyperbola
We can assume that it is similar to that of ellipse's, thus

$$p = a(e^2 - 1)$$

Yet, proof is necessary. The steps are identical.

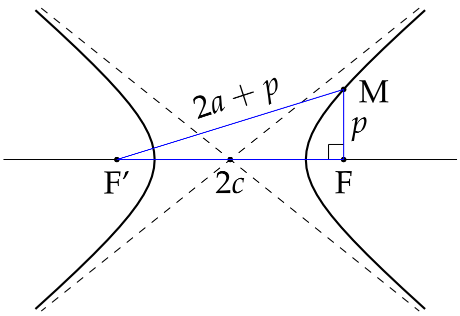


Figure 4: Conic Section

Using pythagorean theorem, we get $(2a + p)^2 = 2c^2 + p^2$ with $a^2 + b^2 = c^2$. Therefore, simplifing lead to the result that:

$$p = a(e^2 - 1)$$

Therefore, we have proven every conic section's semi-latus rectum length.

Area Sweeping Velocity, $\dot{A}$

Area sweeping velocity can be determined by calculating how much of the area is swept after infintely small amount of time dt.

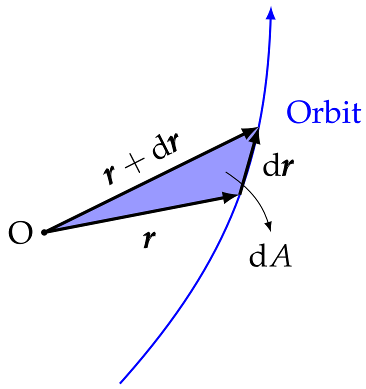


Figure 5: Infinitesimal Area after dt

Let the area swept after infinitely short time dt is dA. When time that has been past is extremely short, curvature of the orbit can be ignored, approximating into linear segment. Then the area becomes triangle, its area can be expressed as:

$$\mathrm{d}A = \frac{1}{2} |\boldsymbol{r} \times \mathrm{d}\boldsymbol{r}|$$

Now differentiate with respect to time. Since cross product is a type of product, product rule holds for vector's cross product.

$$\dot{A} = \frac{1}{2} \left| \boldsymbol{r} \times \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \right| = \frac{1}{2} |\boldsymbol{r} \times \boldsymbol{v}|$$

Using the defintion of specific angular momentum and its application, the equation can be further expanded.

$$\therefore \dot{A} = \frac{1}{2} h = \frac{1}{2} \sqrt{\mu p}.$$

Specific Angular Momentum, $\boldsymbol{h}$

It is already proven, when deriving semi latus rectum, that:

$$h = \sqrt{\mu p}$$

Total Energy, T

Vis-viva equation illustrates the velocity of the orbiting mass. It is expressed as:

$$v^2 = \mu \left(\frac{2}{r} - \frac{1-e^2}{p} \right).$$

During the orbit, only force that acts on the mass is grativtational force. This force causes potential energy and kinetic energy of the mass.

$$T = \frac{1}{2} m_2 v^2 - \frac{\mu m_2}{r}$$

Substitute velocity of mass using vis-viva equation.

$$\begin{aligned} T &= \frac{1}{2} m_2 \mu \left(\frac{2}{r} - \frac{1-e^2}{p} \right) - \frac{\mu m_2}{r} \\ &= \frac{\mu m_2}{2} \cdot \frac{1-e^2}{p} \end{aligned}$$

Final step is to substitute eccentricity that is previously determined by each of the conic sections. After that, total energy of the mass will derived as what table shows.

Velocity Squared, v^2

Mentioned previously, vis-viva equation stresses that:

$$v^2 = \mu \left(\frac{2}{r} - \frac{1-e^2}{p} \right).$$

Let's prove this equation. In arbitrary time, velocity is:

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \equiv \boldsymbol{v} = v_r \hat{\boldsymbol{r}} + v_\theta \hat{\boldsymbol{\theta}}$$

Here,

$$\boldsymbol{r} = (r \sin\theta, r\cos\theta).$$

Differentiating will lead to this result:

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = (\dot{r}\sin\theta + r\cos\theta \cdot \dot{\theta}, \dot{r}\cos\theta - r\sin\theta \cdot \dot{\theta})$$

Since we are derive the velocity sqaured, dot products will lead to that

$$\begin{aligned} v^2 &= \boldsymbol{v} \cdot \boldsymbol{v} \\ &= (\dot{r}\sin\theta + r\cos\theta \cdot \dot{\theta})^2 + (\dot{r}\cos\theta - r\sin\theta \cdot \dot{\theta})^2 \\ &= \dot{r}^2 \sin^2\theta + 2r\dot{r}\sin\theta\cos\theta \cdot \dot{\theta} + r^2\cos^2\theta \cdot \dot{\theta}^2 + \dot{r}^2\cos^2\theta - 2r\dot{r}\sin\theta\cos\theta \cdot \dot{\theta} + r^2\sin^2\theta \cdot \dot{\theta}^2 \\ &= \dot{r}^2(\sin^2\theta + \cos^2\theta) + r^2\dot{\theta}^2(\sin^2\theta + \cos^2\theta)^2 \\ &= \dot{r}^2 + r^2\dot{\theta}^2 \end{aligned}$$

Classification is necessary for $\dot{r}$ and $\dot{\theta}$. $\dot{r}$ can be obtained by differentiating basic equation of motion by time.

$$\begin{aligned}
\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{p}{1+e\cos\theta}\right] \\
&= \frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}\theta}\left[\frac{p}{1+e\cos\theta}\right] \\
&= -\frac{-ep\sin\theta}{(1+e\cos\theta)^2}\dot{\theta} \\
&= \frac{ep\sin\theta}{(1+e\cos\theta)^2}\dot{\theta}
\end{aligned}$$

Recall the velocity of the area segment swept.

$$\dot{A} = \frac{1}{2}h = \frac{1}{2}rv_\perp = \frac{1}{2}r\cdot r\dot{\theta}$$

$$\therefore h = r^2\dot{\theta}$$

Simplifying by squaring each side,

$$\mu p = r^4\dot{\theta}^2, \quad \text{thus} \quad r^2\dot{\theta}^2 = \frac{\mu p}{r^2}.$$

Now, substitute every thing we've got so far.

$$\begin{aligned}
v^2 &= \dot{r}^2 + r^2\dot{\theta}^2 \\
&= \frac{e^2p^2\sin^2\theta}{(1+e\cos\theta)^4}\cdot\dot{\theta}^2 + \frac{\mu p(1+e\cos\theta)^2}{p^2} \\
&= \frac{e^2p^2\sin^2\theta}{(1+e\cos\theta)^4}\cdot\frac{\mu p(1+e\cos\theta)^4}{p^4} + \frac{\mu(1+e\cos\theta)^2}{p} \\
&= \frac{\mu e^2\sin^2\theta}{p} + \frac{\mu(1+e\cos\theta)^2}{p} \\
&= \frac{\mu}{p}\left(e^2\sin^2\theta + 1 + 2e\cos\theta + e^2\cos^2\theta\right) \\
&= \frac{\mu}{p}\left(e^2 - 1 + 2 + 2e\cos\theta\right) \\
&= \mu\left(\frac{e^2-1}{p} + \frac{2+2e\cos\theta}{p}\right) \\
&= \mu\left(\frac{2}{r} - \frac{1-e^2}{p}\right)
\end{aligned}$$

Therefore, proof for vis-viva equation is done. Substituting known information about eccentricity or semi latus rectum length will result in the answer from the table.

(2) During (1) from Problem 2, it is already derived that:

$$\begin{cases} \dot{A} = \frac{1}{2}h \\ h^2 = \mu p \end{cases}$$

Simple algebraic transformation of equations will give a following result:

$$4\dot{A} = \mu p$$

Problem 3

Show that the elliptical orbital motion of the celestial body m_2 from Problem 2 satisfies the virial theorem.

Solution

Virial theorem explains the relation between average kinetic energy, average potential energy, and average total energy. This theorem is crucial, due to the fact that if

$$< K > = -\frac{1}{2} < U > = - < T >$$

satisfies for a certain orbit, then the orbit is stable in that they complete continuous periodic motion, such as circular motion, elliptic motion. The goal of this problem is to prove virial theorem holds for mass that continues an elliptical orbit.

$$< K > = \frac{\int_0^P \frac{1}{2} m_2 v^2 \mathrm{d}t}{\int_0^P \mathrm{d}t}$$

The integration can be done by substituting vis-viva equation and Kepler's equation. Vis-viva equation states that velocity squared in elliptic orbit is:

$$v^2 = \mu \left(\frac{2}{r} - \frac{1}{a} \right)$$

However, we are not able to integrate the equation just by using this, since θ isn't linear of time. We need to find different method of integration. Alternative method involves in using the relation between the ellipse and its circumscribed circle. E is defined as eccentric anomaly, simply an angle of circle to a point that is vertically projected from the ellipse. For a better understanding, figure is provided below.

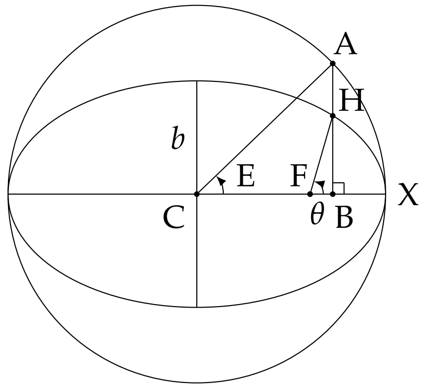


Figure 6: Eccentric Anomaly and True Anomaly

Using the trigonometric relation, we get:

$$\begin{aligned} r\cos\theta &= a\cos E - ae \\ r\sin\theta &= a\sqrt{1-e^2}\sin E. \end{aligned}$$

Therefore,

$$\begin{aligned} r^2 &= a^2\left(\cos E - e\right)^2 + a^2\left(1-e^2\right)\sin^2 E \\ &= a^2\left(\cos^2 E - 2e\cos E + e^2 + \sin^2 E - e^2\sin^2 E\right) \\ &= a^2\left(1 - 2e\cos E + e^2 + e^2\cos^2 E - 1\right) \\ &= a^2\left(1 - e\cos E\right)^2 \end{aligned}$$

Since r is always positive, taking a square root of each side will lead to following:

$$r = a\left(1 - e\cos E\right)$$

Let's use the ellipse's area to find a relation between the time and angle. Let τ be a time that mass is at periapsis. Then, the area FHX (from Figure 6) is expressed as:

$$A = \frac{\pi ab}{P}(t-\tau)$$

We can also consider this area as a subtraction from larger area.

$$\begin{aligned} A &= \frac{b}{a}\left[\text{FAX}\right] \\ &= \frac{b}{a}\left[\text{CAX} - \triangle\text{CAF}\right] \\ &= \frac{b}{a}\left(\frac{1}{2}a^2E - \frac{1}{2}a\cdot ae\sin E\right) \\ &= \frac{1}{2}ab\left(E - e\sin E\right) \end{aligned}$$

$$\therefore \frac{\pi ab}{P}(t-\tau) = \frac{1}{2}ab\left(E - e\sin E\right)$$

Here, mean anomaly (M) is defined as

$$M = \frac{2\pi}{P}(t - \tau)$$

Kepler's equation is a result of this derivation. In conclusion, this equation states that:

$$M = \frac{2\pi}{P}(t - \tau) = E - e \sin E,$$

where M is the mean anomaly and E is the eccentric anomaly. Differentiating both sides by time, we get:

$$\mathrm{d}t \frac{\pi}{P} = (1 - e\cos E)\mathrm{d}E$$

$$\therefore \mathrm{d}t = \frac{P}{\pi}(1 - e\cos E)\mathrm{d}E$$

Therefore, we can calculate mean kinetic energy, using substitution rule.

$$\begin{aligned}
\int_0^P v^2 \mathrm{d}t &= \int_0^{2\pi} \mu \left(\frac{2}{r} - \frac{1}{a}\right) \frac{P}{\pi} (1 - e\cos E)\, \mathrm{d}E \\
&= \int_0^{2\pi} \mu \left(\frac{2}{a(1 - e\cos E)} - \frac{1}{a}\right) \frac{P}{\pi} (1 - e\cos E)\, \mathrm{d}E \\
&= \frac{\mu P}{2\pi a} \int_0^{2\pi} (2 - 1 + e\cos E)\, \mathrm{d}E \\
&= \frac{\mu P}{a}
\end{aligned}$$

Therefore,

$$< K >= \frac{1}{P} \cdot \frac{\mu m_2 P}{2a} = \frac{\mu m_2}{2a}$$

Next step is to calculate the mean of potential energy. Equation has the same format.

$$< U >= \frac{\int_0^P -\frac{\mu m_2}{r} \mathrm{d}t}{\int_0^P \mathrm{d}t}$$

Using the same substitution method from before, we get the following result:

$$\begin{aligned}
< U > &= \frac{1}{P} \int_0^{2\pi} -\frac{\mu m_2}{a\,(1 - e\cos E)} \cdot \frac{P}{2\pi} (1 - e\cos E)\, \mathrm{d}E \\
&= -\frac{\mu m_2}{2\pi a} \int_0^{2\pi} \mathrm{d}E \\
&= -\frac{\mu m_2}{a}
\end{aligned}$$

Here, we got

$$< K >= -\frac{1}{2} < U >$$

The mean of total energy is the sum of mean kinetic energy and mean potential energy. Thus

$$< K >= - < T >= -\frac{1}{2} < U >,$$

and virial theorem satisfies, meaning that we have successful proven that elliptical orbiting motion is indeed periodic and stable.

Problem 4

Find the orbital elements of the celestial body whose ecliptic coordinates and distance are obtained as shown in the table below.

Table 2. Ecliptic Coordinate System Information of a Planet

Date	Ecliptic Longitude	Ecliptic Latitude	Distance to Sun (AU)
July 4th	$181° \ 42' \ 10''$	$1° \ 21' \ 55''$	1.64830
September 2nd	$209° \ 12' \ 08''$	$0° \ 38' \ 05''$	1.59849
November 1st	$238° \ 55' \ 46''$	$-0° \ 18' \ 39''$	1.52637

Solution

This is a similar problem from Problem 1. The purpose is to find all 6 orbital element, but 3 position vectors were provided instead of initial velocity and position vector. Ecliptic coordinate is somewhat similar with spherical coordinate. Figure below is a representation of the ecliptic coordinate; the difference between the ecliptic coordinate and the spherical coordinate is, that in spherical coordinate, latitude is measured from the positive direction of the z axis, while in the ecliptic coordinate, the latitude is measure from the xy plane.

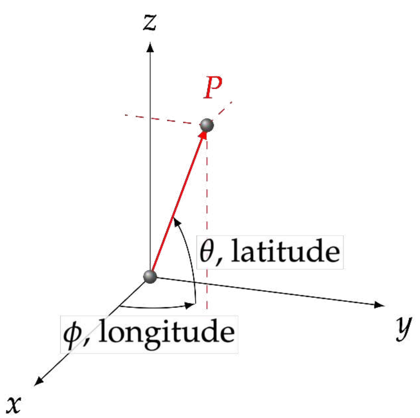


Figure 7: Ecliptic Coordinate

Using the table, let's convert ecliptic coordinate into cartesian coordinate. Single apostrophe means minute, and $1^\circ = 60'$. Double apostrophe means seconds, and $1' = 60''$. Refer to the figure above. Due to geometric relation, the conversion process,

$$(x, y, z) = (r\cos\theta\cos\phi, r\cos\theta\sin\phi, r\sin\theta)$$

Checking if this is valid,

$$\begin{aligned} x^2 + y^2 + z^2 &= r^2\left(\cos^2\theta\left(\cos^2\phi + \sin^2\phi\right) + \sin^2\theta\right) \\ &= r^2\left(\cos^2\theta + \sin^2\theta\right) \\ &= r^2. \end{aligned}$$

Thus, it is valid. I used MATLAB to calculate the values for this section. For every calculation, the code will be provided below.

```
data = [
    181, 42, 10, 1, 21, 55, 1.64830;
    209, 12, 8, 0, 38, 5, 1.59849;
    238, 55, 46, -0, 18, 39, 1.52637;
];

deg2rad = pi/180;

for i = 1:size(data,1)
    longitude = data(i,1) + data(i,2) / 60 + data(i,3) / 3600;
    latitude = data(i,4) + data(i,5) / 60 + data(i,6) / 3600;
    r = data(i,7);

    longitude = longitude * deg2rad;
    latitude = latitude * deg2rad;

    x = r * cos(latitude) * cos(longitude);
    y = r * cos(latitude) * sin(longitude);
    z = r * sin(latitude);

    fprintf('%.8f %.8f %.8f\n', x, y, z);
end
```

After the conversion, it looks like following:

Table 3. Ecliptic Coordinate System Information of a Planet

Day Count	x	y	z
185	-1.64710	-0.0489648	0.0392730
245	-1.39524	-0.779845	0.0177077
305	-0.787738	-1.30737	-0.00828062

6 significant figures

The process of determining orbital elements aren't distinct, but huge single process. Elements are obtained one by one.

Let's refer to the ecliptic coordinate for better understanding.

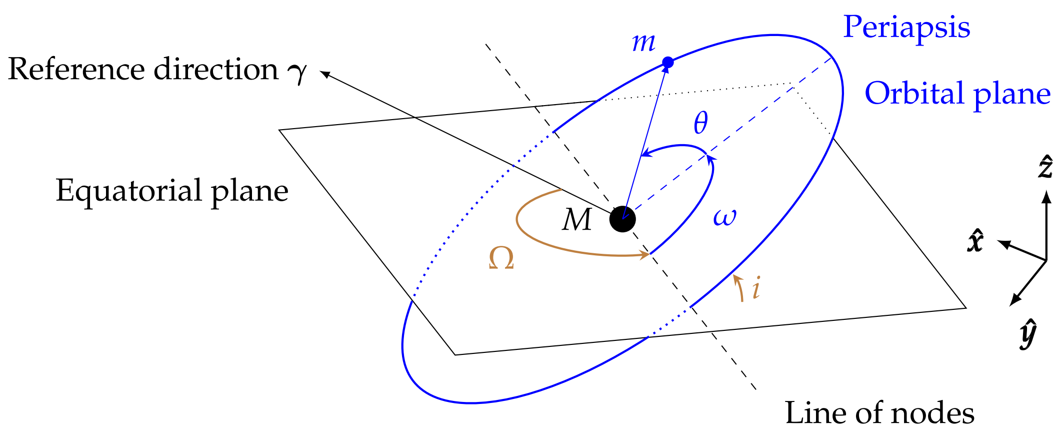


Figure 8: Classical Orbital Elements

Currently, two coordinates are combined in one 3d space, only sharing one origin, the Sun. We want to convert the ecliptic coordinate to the orbital coorinate to see the relation. Since they share the origin, only step to convert it is to proceed rotation. The rotation in 3-dimension is complex, yet converting while fixing one axis can be thought as 2-dimensional rotation, which becomes extremely simple. In 3-dimensional space, the rotation of the x, y axes while fixing z axis is expressed as:

$$R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Let the unit vectors for ecliptic coordinate be $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}$, and the orbital coordinate be $\hat{\boldsymbol{i}}, \hat{\boldsymbol{j}}, \hat{\boldsymbol{k}}$. Then the rotational transformation is expressed as:

$$\begin{pmatrix} \hat{\boldsymbol{x}} \\ \hat{\boldsymbol{y}} \\ \hat{\boldsymbol{z}} \end{pmatrix} = \begin{pmatrix} \cos\Omega & -\sin\Omega & 0 \\ \sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos i & -\sin i \\ 0 & \sin i & \cos i \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{i}} \\ \hat{\boldsymbol{j}} \\ \hat{\boldsymbol{k}} \end{pmatrix}$$

Multiplying inverse matrix on both sides, we get:

$$\begin{pmatrix} \hat{\boldsymbol{i}} \\ \hat{\boldsymbol{j}} \\ \hat{\boldsymbol{k}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos i & \sin i \\ 0 & -\sin i & \cos i \end{pmatrix} \begin{pmatrix} \cos\Omega & \sin\Omega & 0 \\ -\sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{x}} \\ \hat{\boldsymbol{y}} \\ \hat{\boldsymbol{z}} \end{pmatrix}$$

Therefore, we get:

$$\begin{aligned} \hat{\boldsymbol{i}} &= \cos\Omega\hat{\boldsymbol{x}} + \sin\Omega\hat{\boldsymbol{y}} \\ \hat{\boldsymbol{j}} &= (-\cos i \sin\Omega)\hat{\boldsymbol{x}} + (\cos i \sin\Omega)\hat{\boldsymbol{y}} + \sin i\hat{\boldsymbol{z}} \\ \hat{\boldsymbol{k}} &= (\sin i \sin\Omega)\hat{\boldsymbol{x}} + (-\sin i \cos\Omega)\hat{\boldsymbol{y}} + \cos i\hat{\boldsymbol{z}} \end{aligned}$$

The way to utilize this is to applying cross product to any two vector. Then, we can only leave $\hat{\boldsymbol{k}}$. For example, Let's take $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$.

$$\hat{\boldsymbol{k}} = \frac{\boldsymbol{r_1} \times \boldsymbol{r_2}}{|\boldsymbol{r_1} \times \boldsymbol{r_2}|} = \frac{1}{C}\left((y_1 z_2 - z_1 y_2)\hat{\boldsymbol{x}} + (z_1 x_2 + x_1 z_2)\hat{\boldsymbol{y}} + (x_1 y_2 - y_1 x_2)\hat{\boldsymbol{z}}\right)$$

where

$$C = \left((y_1 z_2 - z_1 y_2)^2 + (z_1 x_2 - x_1 z_2)^2 + (x_1 y_2 - y_1 x_2)^2\right)^{1/2}$$

Therefore, we get

$$\begin{aligned} \sin i \sin \Omega &= \frac{1}{C}(y_1 z_2 - z_1 y_2) \\ -\sin i \cos \Omega &= \frac{1}{C}(z_1 x_2 - x_1 z_2) \\ \cos i &= \frac{1}{C}(x_1 y_2 - y_1 x_2) \end{aligned}$$

Dividing $\hat{\boldsymbol{x}}$ term and $\hat{\boldsymbol{y}}$ term,

$$\begin{aligned} \tan \Omega &= -\frac{y_1 z_2 - z_1 y_2}{z_1 x_2 - x_1 z_2} \\ \therefore \Omega &= \arctan\left(-\frac{y_1 z_2 - z_1 y_2}{z_1 x_2 - x_1 z_2}\right) \end{aligned}$$

From $\hat{\boldsymbol{z}}$ term,

$$i = \arccos\left(\frac{1}{C}(x_1 y_2 - y_1 x_2)\right)$$

Since we can rotate between 3 combination of vector choices, I've calculated all 3 combinations and found the mean of for the rigor.

```
x1 = -1.64710;
y1 = -0.0489648;
z1 = 0.0392730;

x2 = -1.39524;
y2 = -0.779845;
z2 = 0.0177077;

x3 = -0.787738;
y3 = -1.30737;
z3 = -0.00828062;

C1 = sqrt((y1 * z2 - z1 * y2)^2 + (z1 * x2 - x1 * z2)^2 + (x1 * y2 - y1 * x2)^2);
Omega1 = atan(-(y1 * z2 - z1 * y2) / (z1 * x2 - x1 * z2));
i1 = acos((1 / C1) * (x1 * y2 - y1 * x2));

C2 = sqrt((y2 * z3 - z2 * y3)^2 + (z2 * x3 - x2 * z3)^2 + (x2 * y3 - y2 * x3)^2);
Omega2 = atan(-(y2 * z3 - z2 * y3) / (z2 * x3 - x2 * z3));
i2 = acos((1 / C2) * (x2 * y3 - y2 * x3));

C3 = sqrt((y1 * z3 - z1 * y3)^2 + (z1 * x3 - x1 * z3)^2 + (x1 * y3 - y1 * x3)^2);
Omega3 = atan(-(y1 * z3 - z1 * y3) / (z1 * x3 - x1 * z3));
```

```
i3 = acos((1 / C3) * (x1 * y3 - y1 * x3));

Omega = mean([Omega1, Omega2, Omega3]);
i = mean([i1, i2, i3]);

Omega = rad2deg(Omega);
i = rad2deg(i);
Omega1 = rad2deg(Omega1);
Omega2 = rad2deg(Omega2);
Omega3 = rad2deg(Omega3);
i1 = rad2deg(i1);
i2 = rad2deg(i2);
i3 = rad2deg(i3);

fprintf('Omega1 = %.6f degrees, i1 = %.6f degrees\n', Omega1, i1);
fprintf('Omega2 = %.6f degrees, i2 = %.6f degrees\n', Omega2, i2);
fprintf('Omega3 = %.6f degrees, i3 = %.6f degrees\n', Omega3, i3);
fprintf('Average Omega = %.6f degrees, Average i = %.6f degrees\n', Omega, i);
```

The result for this code is:

Table 4. Ω and i Determination

Value Type	Ω	i
1st	49.2652°	1.84965°
2nd	49.2606°	1.85006°
3rd	49.2593°	1.84983°
Average	49.2617°	1.84985°

Now, let's consider take into the account that

$$r = \frac{a\left(1-e^2\right)}{1+e\cos\theta}$$

In other words,

$$er\cos\theta = a\left(1-e^2\right)r$$

Let's transform θ into $\omega+\theta-\omega$. Define $u \equiv \omega+\theta$. Then,

$$\cos\theta = \cos u\cos\omega + \sin u\sin\omega$$

Substitute this equation:

$$e\left(\cos u\cos\omega + \sin u\sin\omega\right)r = a\left(1-e^2\right) - r$$

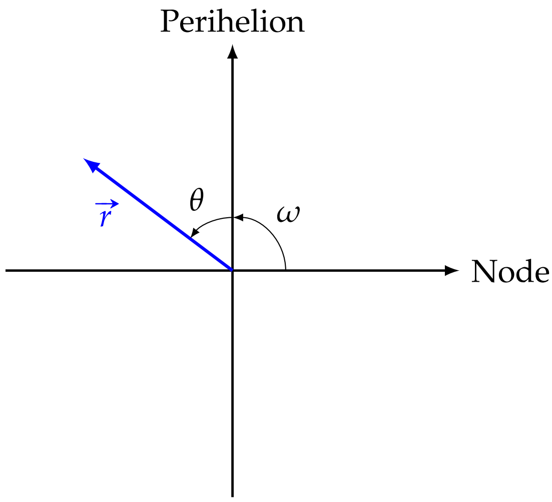


Figure 9: Simplified 2-Dimension Representation

Refer to the figure above. Then, since $u = \omega + \theta$,

$$r \cos u = \boldsymbol{r} \cdot \hat{\boldsymbol{x}}$$
$$r \sin u = \boldsymbol{r} \cdot \hat{\boldsymbol{y}}$$

Substitute this, we get

$$e\left((\boldsymbol{r} \cdot \hat{\boldsymbol{x}}) \cos \omega + (\boldsymbol{r} \cdot \hat{\boldsymbol{y}}) \sin \omega\right) = a\left(1 - e^2\right) - r$$

Applying this equation for 2 different cases, we get a system of equations.

$$e \cos \omega((\boldsymbol{r_1} - \boldsymbol{r_2}) \cdot \hat{\boldsymbol{x}}) + e \sin \omega((\boldsymbol{r_1} - \boldsymbol{r_2}) \cdot \hat{\boldsymbol{y}}) = r_2 - r_1$$
$$e \cos \omega((\boldsymbol{r_1} - \boldsymbol{r_3}) \cdot \hat{\boldsymbol{x}}) + e \sin \omega((\boldsymbol{r_1} - \boldsymbol{r_3}) \cdot \hat{\boldsymbol{y}}) = r_3 - r_1$$

Since this equation is a system of **linear** equations, solving this is just a calculation; the result is:

$$e \cos \omega = \frac{A_1}{D}, \text{ and } e \sin \omega = \frac{A_2}{D}$$

where

$$A_1 = ((r_2 - r_3)\boldsymbol{r_1} + (r_3 - r_1)\boldsymbol{r_2} + (r_1 - r_2)\boldsymbol{r_3}) \cdot \hat{\boldsymbol{y}}$$
$$A_2 = -((r_2 - r_3)\boldsymbol{r_1} + (r_3 - r_1)\boldsymbol{r_2} + (r_1 - r_2)\boldsymbol{r_3}) \cdot \hat{\boldsymbol{x}}$$
$$D = \det\begin{pmatrix} (\boldsymbol{r_1} - \boldsymbol{r_2}) \cdot \hat{\boldsymbol{x}} & (\boldsymbol{r_1} - \boldsymbol{r_2}) \cdot \hat{\boldsymbol{y}} \\ (\boldsymbol{r_1} - \boldsymbol{r_3}) \cdot \hat{\boldsymbol{x}} & (\boldsymbol{r_1} - \boldsymbol{r_3}) \cdot \hat{\boldsymbol{y}} \end{pmatrix}$$

Therefore,

$$e = \frac{\sqrt{A_1^2 + A_2^2}}{D}, \text{ and } \omega = \arctan\left(\frac{A_2}{A_1}\right)$$

By substituting any data value in the previous equations, we get semi-major axis.

$$a = \frac{1}{1 - e^2}\left(r + (\boldsymbol{r} \cdot \hat{\boldsymbol{x}}) \cdot \frac{A_1}{D} + (\boldsymbol{r} \cdot \hat{\boldsymbol{y}}) \cdot \frac{A_2}{D}\right)$$

This process is also calculated using MATLAB coding. The raw code is provided below.

```
x1 = -1.64710;
y1 = -0.0489648;
z1 = 0.0392730;

x2 = -1.39524;
y2 = -0.779845;
z2 = 0.0177077;

x3 = -0.787738;
y3 = -1.30737;
z3 = -0.00828062;

r1 = [x1, y1, z1];
r2 = [x2, y2, z2];
r3 = [x3, y3, z3];

d1 = 1.64830;
d2 = 1.59849;
d3 = 1.52637;

r12 = r1 - r2;
r13 = r1 - r3;

A1 = (d2 - d3) * r1(2) + (d3 - d1) * r2(2) + (d1 - d2) * r3(2);
A2 = -((d2 - d3) * r1(1) + (d3 - d1) * r2(1) + (d1 - d2) * r3(1));

D = det([r12(1) r12(2); r13(1) r13(2)]);

e = sqrt(A1^2 + A2^2) / D;
omega = atan(A2/A1);
if omega < 0
    omega = omega + 2 * pi;
end

omega_deg = rad2deg(omega);

fprintf('Eccentricity = %.6f\n', e);
fprintf('Argument of perihelion = %.6f degrees\n', omega_deg);

a1 = (1 / (1 - e^2)) * (d1 + (r1(1)) * (A1 / D) + (r1(2)) * (A2 / D));
fprintf('Semi-major axis for r1 = %.6f\n', a1);

a2 = (1 / (1 - e^2)) * (d2 + (r2(1)) * (A1 / D) + (r2(2)) * (A2 / D));
fprintf('Semi-major axis for r2 = %.6f\n', a2);

a3 = (1 / (1 - e^2)) * (d3 + (r3(1)) * (A1 / D) + (r3(2)) * (A2 / D));
fprintf('Semi-major axis for r3 = %.6f\n', a3);

a = mean([a1, a2, a3]);
fprintf('Mean a = %.6f\n', a);
```

The result for this is presented below:

$$e = 0.0934307, \;\; \omega = 335.413^\circ, \;\; a = 1.52357 \text{ AU}$$

Surprisingly, even though semi-major axis was measured using the mean of 3 different combination of calculation, the values were identical for every case.

For the final step, we can calculate the time of passing perihelion, τ. Using the eccently anomaly equation, we get:

$$r = a(1 - e\cos E)$$
$$\therefore E = \arccos\left(\frac{a-r}{ae}\right)$$

Combining with the Kepler's equation,

$$E - e\sin E = M = \frac{2\pi}{P}(t-\tau)$$
$$\therefore \tau = t - \frac{P}{2\pi}(E - e\sin E)$$

Here is the code for data processing. τ depends on the reference value. For simplicity, P is calculated using Kepler's 3rd law, since the semi-major axis is determined. Calculations are done using the following MATLAB code.

```
a = 1.523566;
e = 0.093431;
P = a^(3/2) * 365.242;

t1 = 185;
t2 = 245;
t3 = 305;

r1 = 1.64830;
r2 = 1.59849;
r3 = 1.52637;

E1 = acos((a - r1) ./ (a * e));
E2 = acos((a - r2) ./ (a * e));
E3 = acos((a - r3) ./ (a * e));

tau1 = t1 - (P / (2 * pi)) * (E1 - e * sin(E1));
tau2 = t2 - (P / (2 * pi)) * (E2 - e * sin(E2));
tau3 = t3 - (P / (2 * pi)) * (E3 - e * sin(E3));

fprintf('t = 185, E = %.6f, tau = %.6f\n', E1, tau1);
fprintf('t = 185, E = %.6f, tau = %.6f\n', E2, tau2);
fprintf('t = 185, E = %.6f, tau = %.6f\n', E3, tau3);
```

The result is:

Table 5. t, E, and τ Determination

Value Type	t	E	τ
1st	185	2.63883	-98.5520
2nd	245	2.12509	21.3733
3rd	305	1.59050	141.341
Average	N/A	2.11814	21.3876
Standard Deviation	N/A	0.524206	119.947

N/A: Not Appicable

Since there are several angles that make the same value of cosine, i.e.,

$$\cos\theta = \cos(2\pi - \theta).$$

Therefore, we can plug in $2\pi - \theta$ for the calculation. Modify the code a little, we get:

Table 6. t, E, and τ Determination With New Method

Value Type	t	E	τ
1st	185	3.64435	-218.315
2nd	245	4.15810	-218.240
3rd	305	4.69269	-218.208
Average	N/A	4.16504	-218.255
Standard Deviation	N/A	0.524206	0.054698

N/A: Not Appicable

Therefore, τ is determined. Since τ represents day, it should be integer. Rounding it into an integer, we get $\tau = -218$, and this is May 28th, and the year before.

To conclude:

Table 7. Results for Orbital Elements

i	Ω	e	a	ω	τ
1.84985°	49.3627°	0.0934307	1.52357 AU	335.413°	-218

6 significant figures with τ being the integer

This orbit seems to be identical to Mars's orbit. For comparison,

Table 8. Orbital Elements

Data Type	i	Ω	e	a	ω	τ
Experimental	1.84985°	49.3627°	0.0934307	1.52357 AU	335.413°	-218
Accepted Value	1.85061°	49.5785°	0.0934123	1.52366 AU	336.040°	N/A
Percent Error	0.0410%	0.4393%	0.0198%	0.0059%	0.1863%	N/A

N/A: Not Available, Data from NASA: https://nssdc.gsfc.nasa.gov/planetary/factsheet/marsfact.html

The results turn out to be extremely correct, it can be said that the calculations are all done precisely.

Problem 5

As shown in the figure below, let's assume that when the Earth starts from E_1, rotates once, and comes to the location of E_2, the probe launched from E_1 orbits around Mars and then meets the Earth at E_2. Find the semi-major axis, eccentricity, and return time of this probe to Earth. Here, θ equals to 30°, and ignore the time the probe stays on Mars.

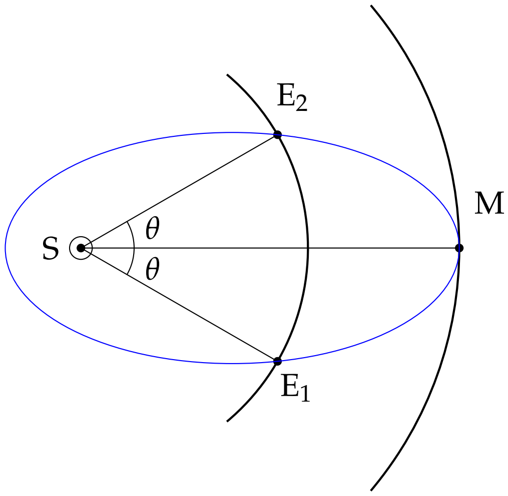


Figure 10: Diagram for Problem 5

Solution

On the whole process, the Earth has rotated 7/6 revolutions. Assume that the Earth's orbit is a complete circle. (Since the actual eccentricity for the Earth's orbit is 0.0167, it can be approximated as a circle without causing a huge error.) The distance from the Sun and Mars cannot be approximated because it fluctuates a lot. The minimum distance is 1.3814 AU, while the maximum distance is 1.6660 AU. It is not plausible to set the distance to Mars as constant. Without knowing the distance to the Mars, we have to find out specific information about this orbit.

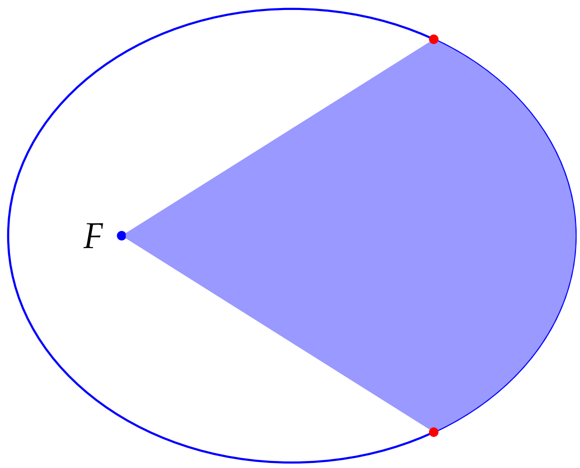


Figure 11: Area Swept by the Probe

Refer to the figure above. This is the area swept by the probe through the entire trip. The reason why this area is important is because of the Kepler's 2nd law of planetary motion, saying that planet (rigorously a segment created by planet and a sun) sweeps out equal areas during equal intervals of time. Therefore, the entire area of the ellipse indicates period of the entire trace.

The equation for the ellipse is expressed as:

$$r = \frac{p}{1 - e\cos\theta}$$

We have to find two unknown constant: p and e. One information we have is that when $\theta = 30^\circ$, it reaches the Earth, thus $r = 1$. The unit for r is AU, Astronomical Unit. Substitute this, we get:

$$p = 1 - \frac{\sqrt{3}}{2}e$$

The final information we have is the Kepler's 3rd law. According to this law,

$$P^2 \propto a^3$$

Combining two laws of Kepler, we can know that area colored above is equivalent of 7/6 years. And the entire area is equivalent to the period of this elliptical path, and this period is determined by the semi-major axis, a. Here, semi-major axis is also a function of e.

$$a = \left.\frac{p}{1 - e\cos\theta}\right|_{\theta=0} = \frac{1 - \frac{\sqrt{3}}{2}e}{1 - e}$$

This means that after we calculate the area of ellipse, we can find every information we need. Let's refer back to the step of how to calculate infinitesimal area.

$$\mathrm{d}A = \frac{1}{2}\left|\boldsymbol{r} \times \mathrm{d}\boldsymbol{r}\right|$$

Here,

$$\mathrm{d}\boldsymbol{r} = \dot{r}\hat{\boldsymbol{r}} + r\hat{\boldsymbol{\theta}}$$

Since $\hat{\boldsymbol{r}}$ is always parallel to $\boldsymbol{r}$ and $\hat{\boldsymbol{\theta}}$ is always perpendicular to $\boldsymbol{r}$,

$$\boldsymbol{r} \times \mathrm{d}\boldsymbol{r} = r^2 \mathrm{d}\theta$$

Therefore, the area of portion of the ellipse can expressed as:

$$A = \int_{\theta_1}^{\theta_2} r^2 \mathrm{d}\theta = \int_{\theta_1}^{\theta_2} \left(\frac{p}{1 - e\cos\theta} \right)^2 \mathrm{d}\theta$$

Here, the portion of colored area (on the figure above) is I_1 / I_2, where

$$I_1 = \int_{-\pi/6}^{\pi/6} \left(\frac{p}{1 - e\cos\theta} \right)^2 \mathrm{d}\theta$$

$$I_2 = \int_{0}^{2\pi} \left(\frac{p}{1 - e\cos\theta} \right)^2 \mathrm{d}\theta$$

According to Kepler's 3rd law,

$$P = \left(\frac{1 - \frac{\sqrt{3}}{2}e}{1 - e} \right)^{3/2} \text{[years]}$$

Since it took $7/6$ years for completing the orbit, we get

$$\frac{I_1}{I_2} \cdot P = \frac{7}{6}$$

The integration for the ellipse is not simple, and even though we finish the integral, we cannot compute and find the solution using algebraic method. Therefore, we will use numerical method to interpret and find the solution. To provide the raw solution for the easiest form of integration is:

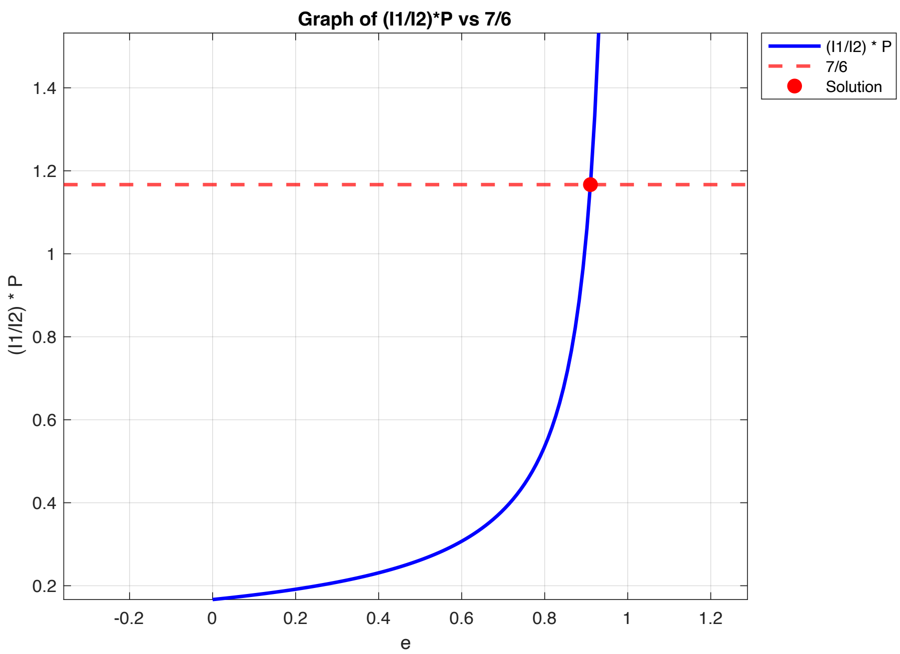
$$\int \left(\frac{1}{1 - k\cos x} \right)^2 \mathrm{d}x = \frac{k \sin x}{(k^2 - 1)(k\cos x - 1)} + \frac{2\tanh^{-1}\left(\frac{(k+1)\tan(\pi/2)}{\sqrt{k^2-1}} \right)}{(k^2 - 1)^{3/2}} + C$$

Therefore, numerical method might the only (possible) solution for this problem. For this process, I have used MATLAB for the calculation. Using computer, it calculates which value should e be, to satisfy our equation.

Here is the code for MATLAB:

```
r2 = @(x,y) ((1 - sqrt(3)/2 * x) ./ (1 - x * cos(y))).^2;

I1 = @(x) integral(@(y) r2(x,y), -pi/6, pi/6);
I2 = @(x) integral(@(y) r2(x,y), 0, 2*pi);

P = @(x) ((1 - sqrt(3)/2 * x) / (1 - x^2))^(3/2);

equation = @(x) (I1(x) / I2(x)) * P(x) - 7/6;
solution = fzero(equation, 0.5);

fprintf('The value of x that satisfies the equation is: %.6f\n', solution);

X = linspace(0, 0.93, 100);

Y = arrayfun(@(x) (I1(x) / I2(x)) * P(x), X);

figure('Position', [100, 100, 600, 400]);
plot(X, Y, 'b-', 'LineWidth', 2);
hold on;
yline(7/6, 'r--', 'LineWidth', 2);
plot(solution, (I1(solution) / I2(solution)) * P(solution), 'ro', 'MarkerSize', 8,
    'MarkerFaceColor', 'r');

title('Graph of (I1/I2)*P vs 7/6');
xlabel('e');
ylabel('(I1/I2) * P');
legend({'(I1/I2) * P', '7/6', 'Solution'}, 'Location', 'northeastoutside');
grid on;
axis equal;

saveas(gcf, 'Matlab.pdf');
```

This code draws a graph of $P \cdot I_1/I_2$ and finds a intercept with $y = 7/6$. The figure we got after running the code is given like following:



The result we get is:

$$e = 0.91024$$

It indicates that the ellipse we obtained is nearly a parabola, highly stretched ellipse. To draw the path for the probe on the 2D plane, we get the following figure:

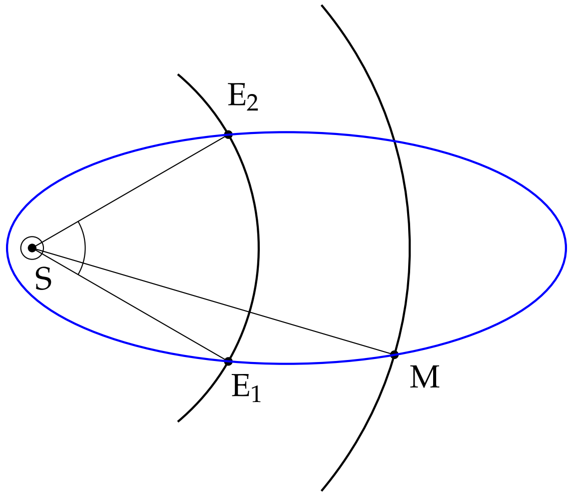


Figure 12: Diagram for Problem 5

Here,

$$p = 1 - \frac{\sqrt{3}}{2}e \Bigg|_{e=0.91024} = 0.21171$$

$$r = \frac{0.21171}{1 - 0.91024\cos\theta}$$

$$a = \frac{p}{1-e^2} = 2.35862 \text{ AU}$$

Surprisingly, the figure that came out after the calculation seems to be quite different from what is given at the problem. How can I ensure this?

$$r_a = \frac{p}{1-e} = 2.35862 \text{ AU} > 1.66660 \text{ AU}$$

Since apogee of the probe is way bigger than the Mars's, we can guarantee that it will not be tangential to the Mars's orbit. Even though problem has given a different figure, by applying Kepler's law of planetary motion and solving it using numerical method, it is proven that Figure 12 is the correct figure. To say that the probe touches Mars's orbit in one point (tangentially), different θ should be provided.

Assignment 3

Stars and Universe
June 15, 2025
23-119 ONE CHOI

Problem 1

Explain an example of orbital resonance in the solar system.

Solution

Well-known examples of orbital resonance often include the moons of the Saturn or the Jupiter. However, even the planets show the orbital resonance. The two planets are the Neptune and the Pluto. In our solar system, they interact with gravitational force, causing the orbital resonance.

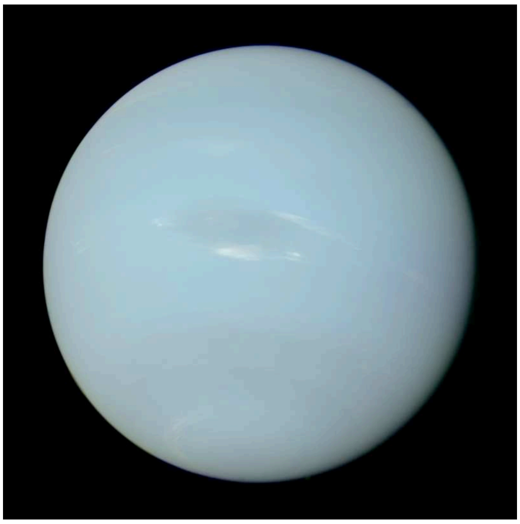


Figure 1: Neptune

Figure 2: Pluto

The essential information for these planets are known as follows:

[0]Image of the Neptune is captured by Voyager 2 Mission on 1989. For the Pluto, the image is captured by New Horizons Mission in 2015. Since this mission is recently implemented, we got the clearest PLuto image ever. All photos are from NASA.

Table 1. Basic Data for Neptune and Pluto

Characteristics	**Neptune ♆**	**Pluto ♇**
Body Type	Planet	Dwarf Planet
Orbital Period (years)	164.791	247.921
Mass (kg)	1.024×10^{26}	1.303×10^{22}
Resonance Ratio	2	3
Average Orbital Radius (AU)	30.07	39.48
Orbital Eccentricity	0.008678	0.248807
Diameter (km)	49,244	2,376
Tilt of Axis (degrees)	28.32°	122.53°

♆: Neptune, ♇: Pluto

We can see that these two planets are orbiting on a period of **integer ratio**. While it is not exactly an integer ratio, it is approximated.

$$164.791 : 247.912 = 2.0000 : 3.0088 \approx 2 : 3$$

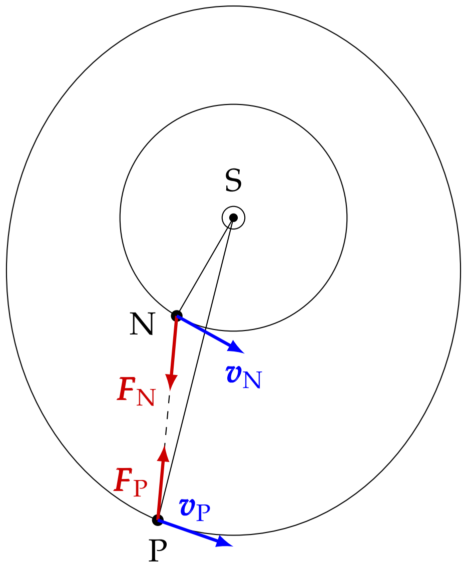


Figure 3: Diagram for Problem 5

Refer to the figure above. As shown there, due to a pair of gravitational force between the Neptune and the Pluto, eccentricity orbital resonance occur, with

$$P_\mathrm{N} : P_\mathrm{P} = 2 : 3 \text{ and } \omega_\mathrm{N} : \omega_\mathrm{P} = 3 : 2,$$

where ω represents angular velocity. The reason why this can happen to the planets is because the Neptune and the Pluto are way too far away from the Sun, more than 30 AU in distance, making them less influenced by the gravitational field of the Sun. In other words, adjacent planets were able to cause influence to their orbits.

Problem 2

Find the Lagrangian point L2 or L3.

Solution

Lagrange point is a equilibrium point on a space under the influence of the gravitational pull of a star and a planet. In a simple system of the Sun and the Earth, 5 Lagrange points exists. 5 points are plotted below

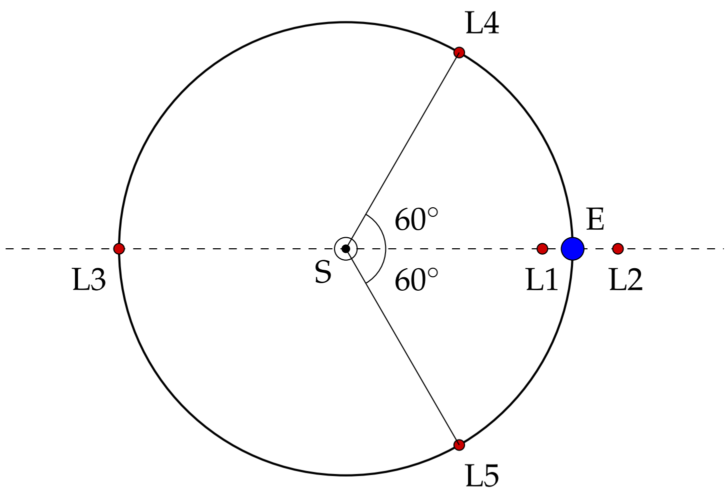


Figure 4: Diagram for Problem 5

We'll be finding both Lagrange point 2 and 3.

L2

The simple way of calculating this involves a Newtonian mechanics. Refer to the figure below.

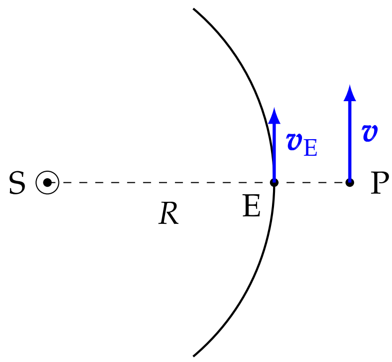


Figure 5: Diagram for Problem 5

The condition for the Lagrange points was that they need to be in the gravitational equilibrium state. Only forces working on this system is gravitational attraction. Analyzing forces, we get that

$$\frac{GMm}{R^2} = \frac{mv_E^2}{R}$$

since the Earth should have a circular motion. Here, G is a gravitational constant, M is a mass of the Sun, m is a mass of the Earth. In addition, let m' be the mass of the object. Simplifying the speed of the Earth:

$$v_E = \sqrt{\frac{GM}{R}}$$

Since it is on a equilibrium state, the position should stay constant no matter the Earth is rotating. The velocity is expressed as $v = r\omega$. Therefore, the radial velocity should be proportional to the distance from the Sun. Let the distance from the Sun to the object be x, i.e., $\overline{\mathrm{SP}} = x$. Then, the velocity of the object v is expressed as:

$$v = \frac{x}{R}\sqrt{\frac{GM}{R}}.$$

And finally, the centripetal force for the object's revolution is the gravitational pull from the Sun and the Earth.

$$\frac{m'}{x}\frac{x^2}{R^2}\frac{GM}{R} = \frac{Gmm'}{(x-R)^2} + \frac{GMm'}{x^2}$$

Simplifying this equation, we are left with:

$$\frac{M}{R^3}x = \frac{m}{(x-R)^2} + \frac{M}{x^2}$$

The equation cannot simply be solved by hands. Thus I used MATLAB program to find the zeros for this equation. The raw code is provided below:

```
M = 1.988416e30;
m = 5.9722e24;
R = 1;

equation = @(x) (M/R^3)*x - (m/(x-R)^2) - (M/x^2);

guess = 1.5*R;
solution = fzero(equation,guess);

fprintf('x = %.6f\n', solution);
```

The result is given as:

$$x = 1.010037 \text{ AU}$$

From the Earth, this distance is approximately 4 times the distance to the Moon.

L3

3rd Lagrange point can be determined in a similar manner. The velocity of the Earth's orbit is identical. The only different is the relative position of the object.

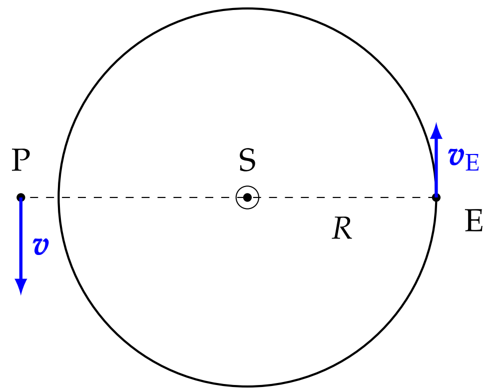


Figure 6: Diagram for Problem 5

To write this situation into equation:

$$\frac{m'}{x}\frac{x^2}{R^2}\frac{GM}{R} = \frac{Gmm'}{(x+R)^2} + \frac{GMm'}{x^2}.$$

Simplifying, we get:

$$\frac{M}{R^3}x = \frac{m}{(x+R)^2} + \frac{M}{x^2}$$

```
M = 1.988416e30;
m = 5.9722e24;
R = 1;

equation = @(x) (M/R^3)*x - (m/(x+R)^2) - (M/x^2);

guess = 1.5*R;
solution = fzero(equation,guess);

fprintf('x = %.6f\n', solution);
```

The solution can be obtained the same way.

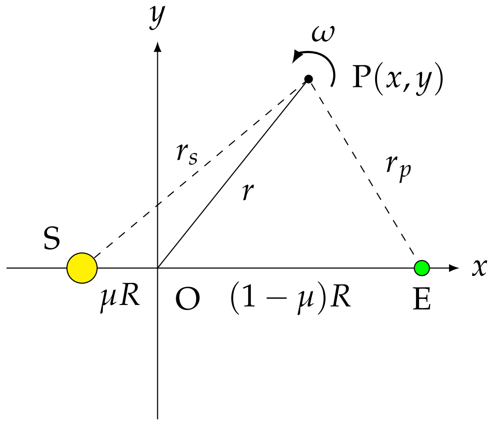
$$x = 1.000000 \text{ AU}$$

This may look like x is exactly equal to 1, but it is because the mass of the Earth is significantly small compared to the Sun. In conclusion, the Lagrange point's location is determined.

Precise method

However, the answer above containes a lot of approximations and assumptions. In addition to that, a lot of physical meanings are missing. I though it would be significant to find more general and precise way of calculating.

This time, we will not assume that the Sun is the center of mass of the entire system. And on an arbitrary position, the object rotates in respect to the center of mass. This time let the center of mass be the origin.



This time, the ratio of the mass of the Earth and the mass of the Sun be μ. If we let the mass of the Sun be M, then the mass of the Earth will be μM. Generally, $\mu < 0$. The mass of the object, m, is negligibly small, thus the position of the figure above holds, and the origin is the center of mass. Setting a **non-inertial frame** of the object, we get the potential energy as:

$$U = -\frac{GMm}{r_s} - \frac{G\mu Mm}{r_p} - \frac{1}{2}mv^2$$

Considering the ratio of the distance from the center of mass, we get:

$$v = \frac{r}{(1-\mu)R}v_E$$

One thing to notice is that v_E is not identical from the previous method.

$$\frac{G\mu M^2}{R^2} = \mu M \frac{v_E^2}{(1-\mu)R}$$

$$\therefore v_E = \sqrt{\frac{GM}{(1-\mu)R}}$$

Substitute, we get:

$$U = -\frac{GMm}{r_s} - \frac{G\mu Mm}{r_p} - \frac{1}{2}m\frac{GMr^2}{((1-\mu)R)^3}$$

The reason why there is a kinetic energy term inside a potential energy equation, is because the frame is set as non-inertial frame, it rotates in circle, affected by centripetal force. Different kinds of r can be determined using x and y.

$$r_s = \sqrt{(x+\mu R)^2 + y^2}$$

$$r_p = \sqrt{((1-\mu)R - x)^2 + y^2}$$
$$r = \sqrt{x^2 + y^2}$$

Substituting these, then the potential energy is determined. Since the potential energy is directly proportional to the mass of the object, we will look at specific potential enregy, which is potential energy divided by the mass of the object, i.e., U/m.

$$\frac{U}{m} = -\frac{GM}{\sqrt{(x+\mu R)^2 + y^2}} - \frac{G\mu M}{\sqrt{((1-\mu)R - x)^2 + y^2}} - \frac{1}{2}\frac{(x^2+y^2)GM}{(1-\mu)R^3}$$

Using MATLAB, we are able to plot a 3D graph, consisting x axis, y axis, and U/m axis. The raw code is:

```
G = 1;
M = 5;
k = 0.1;
R = 1;

[x,y] = meshgrid(linspace(-2, 2, 2000), linspace(-2, 2, 2000));

r1 = sqrt((x + k*R).^2 + y.^2);
r2 = sqrt((x - (1-k)*R).^2 + y.^2);

U_m = - (G*M ./ r1) - (k*G*M ./ r2) - 0.5 * (G*M / ((1-k) * R^3)) * (x.^2 + y.^2);
U_m(U_m < -17) = NaN;

figure;
surf(x,y,U_m);
shading interp;
colorbar;
title('3D Plot of U/m');
xlabel('x');
ylabel('y');
zlabel('U/m');
view(50,45);

hold on;
plot3(-k*R, 0, -8, 'ro', 'MarkerSize', 10, 'MarkerFaceColor', 'r'); % sun
plot3((1-k)*R, 0, -8, 'bo', 'MarkerSize', 7, 'MarkerFaceColor', 'b'); % earth

plot3(-1.04, 0, -8.5, 'bo', 'MarkerSize', 5, 'MarkerFaceColor', 'g');
plot3(1.25, 0, -9.3, 'bo', 'MarkerSize', 5, 'MarkerFaceColor', 'g');
plot3(0.6, 0, -9.5, 'bo', 'MarkerSize', 5, 'MarkerFaceColor', 'g');
plot3(0.55, -0.788, -7.9, 'bo', 'MarkerSize', 5, 'MarkerFaceColor', 'g');
plot3(0.55, 0.788, -7.9, 'bo', 'MarkerSize', 5, 'MarkerFaceColor', 'g');

hold off;
```

This code plots a 3D graph of potential distribution by the position. Since substituting real-world value leads to extreme gradient of the graph, I used simple constants to make the graph easier interpret, such as using $G = 1$, $M = 1$, and $\mu = k = 0.1$ The result is as follows.

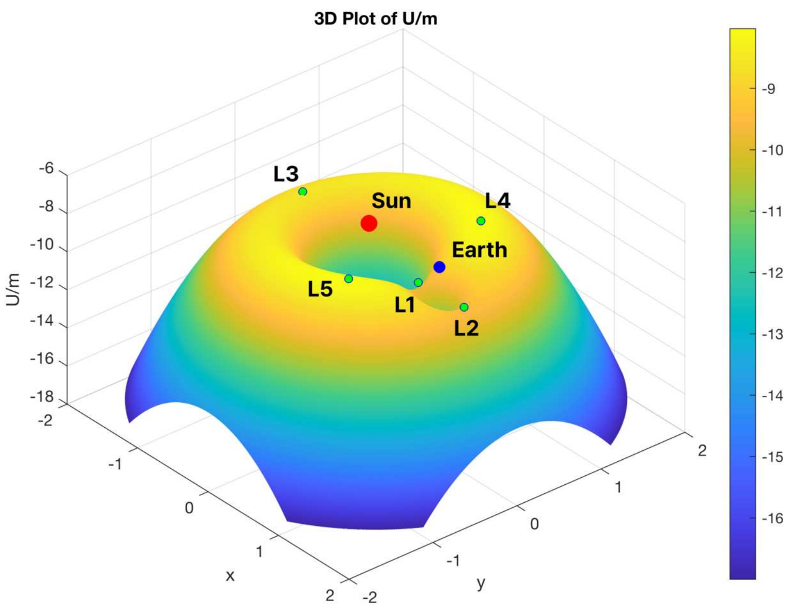


Figure 7: 3D Plot of Potential Distribution

Here, you can notice that all 5 Lagrangian points are at equilibrium state. I.e.,

$$\nabla U = 0.$$

One more thing to notice is the second derivative of the potential distribution. All Lagrangian points are located such at small change in the position will result the object to escape the equilibrium state. When

$$\nabla^2 U < 0,$$

It is called unstable equilibrium; slight displacement will lead not make it back to the equilibrium point.

Another way of plotting this graph is using contour graph. By using the colormap on 2D plane, which is easier to interpret, we can approximately notice which area has the same potential. The code looks similar, except it involves drawing a contour graph.

```
G = 1;
M = 5;
k = 0.1;
R = 1;

[x,y] = meshgrid(linspace(-2, 2, 2000), linspace(-2, 2, 2000));

r1 = sqrt((x + k*R).^2 + y.^2);
r2 = sqrt((x - (1-k)*R).^2 + y.^2);

U_m = - (G*M ./ r1) - (k*G*M ./ r2) - 0.5 * (G*M / ((1-k) * R^3)) * (x.^2 + y.^2);
U_m(U_m < -17) = NaN;

figure;
```

```
contourf(x, y, U_m, 20);
colorbar;
title('Contour Plot of U/m');
xlabel('x');
ylabel('y');

hold on;
plot(-k*R, 0, 'ro', 'MarkerSize', 10, 'MarkerFaceColor', 'r'); % sun
plot((1-k)*R, 0, 'bo', 'MarkerSize', 7, 'MarkerFaceColor', 'b'); % earth

plot(-1.04, 0, 'bo', 'MarkerSize', 5, 'MarkerFaceColor', 'g');
plot(1.25, 0, 'bo', 'MarkerSize', 5, 'MarkerFaceColor', 'g');
plot(0.6, 0, 'bo', 'MarkerSize', 5, 'MarkerFaceColor', 'g');
plot(0.7, -0.788, 'bo', 'MarkerSize', 5, 'MarkerFaceColor', 'g');
plot(0.7, 0.788, 'bo', 'MarkerSize', 5, 'MarkerFaceColor', 'g');

hold off;
```

The contour graph looks like following:

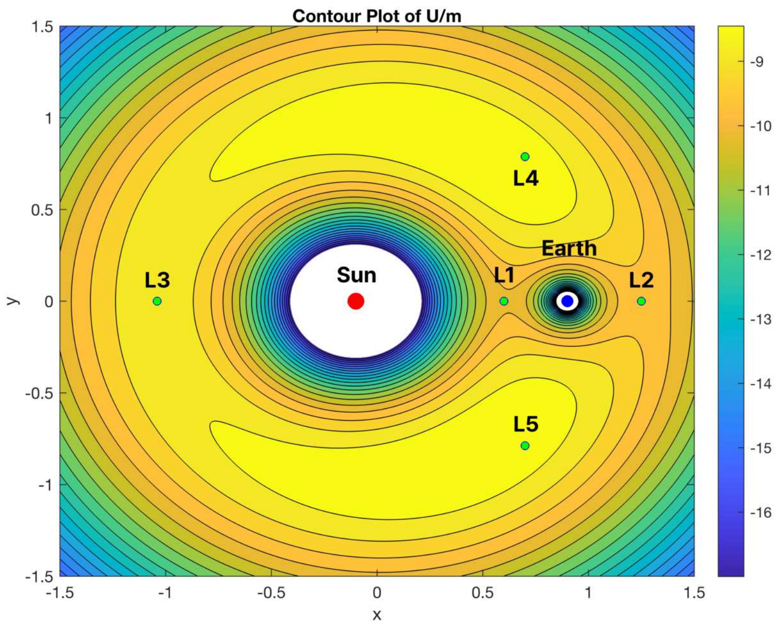


Figure 8: Contour Graph of Potential Distribution

Therefore, we successfully determined and interpreted all 5 Lagrange points using generalized equation. Specifically, if you want to find the L2 and L3, you can simply substitute $y = 0$, $R = 1$, and differentiate with respect to x. The x value that satisfies this is the Lagrangian point.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{GM}{|x+\mu|} - \frac{G\mu M}{|1-\mu-x|} - \frac{x^2 GM}{2(1-\mu)}\right) = 0$$

In this equation, GM appears as the common factor for all terms. In differentiation, common constant multiplied doesn't affect the derivative to be 0. Therefore, the problem simplifies into solving:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{1}{|x+\mu|}-\frac{\mu}{|1-\mu-x|}-\frac{x^2}{2(1-\mu)}\right)=0$$

First, we need to understand how differentiating $|x|$ works. We should think a way that controls the sign of the derivative. The method looks like following:

$$\frac{\mathrm{d}}{\mathrm{d}x}|x|=\frac{x}{|x|}$$

Then, when $x<0$, $\frac{\mathrm{d}}{\mathrm{d}x}|x|=-1$ and when $x>0$, $\frac{\mathrm{d}}{\mathrm{d}x}|x|=1$. Applying the chain rule, we get:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{1}{|x+\mu|}-\frac{\mu}{|1-\mu-x|}-\frac{x^2}{2(1-\mu)}\right)\\ =&\frac{1}{(x+\mu)^2}\cdot\frac{\mathrm{d}}{\mathrm{d}x}|x+\mu|+\frac{\mu}{(1-\mu-x)^2}\cdot\frac{\mathrm{d}}{\mathrm{d}x}|1-\mu-x|-\frac{x}{1-\mu}\end{aligned}$$

Finishing the calculation, we are left with:

$$\frac{1}{(x+\mu)\,|x+\mu|}+\frac{\mu}{(x+\mu-1)\,|x+\mu-1|}-\frac{x}{1-\mu}=0$$

Using MATLAB, every x that satisfies this equation is determined.

```
syms x
mu = 3.004297e-6;

% x + mu >= 0 and x + mu - 1 >= 0
equ1 = 1 / ((x + mu) * (x + mu)) + mu / ((x + mu - 1) * (x + mu - 1)) - x / (1 - mu);
sol1 = solve(equ1 == 0, x);

% x + mu < 0 and x + mu - 1 >= 0
equ2 = 1 / ((x + mu) * -(x + mu)) + mu / ((x + mu - 1) * (x + mu - 1)) - x / (1 - mu);
sol2 = solve(equ2 == 0, x);

% x + mu >= 0 and x + mu - 1 < 0
equ3 = 1 / ((x + mu) * (x + mu)) + mu / ((x + mu - 1) * -(x + mu - 1)) - x / (1 - mu);
sol3 = solve(equ3 == 0, x);

% x + mu < 0 and x + mu - 1 < 0
equ4 = 1 / ((x + mu) * -(x + mu)) + mu / ((x + mu - 1) * -(x + mu - 1)) - x / (1 - mu);
sol4 = solve(equ4 == 0, x);

sol1 = double(sol1);
sol2 = double(sol2);
sol3 = double(sol3);
sol4 = double(sol4);

solution1 = sol1(imag(sol1) == 0 & sol1 >= -mu & sol1 >= (1 - mu)); % x + mu >= 0 and x + mu - 1 >= 0
solution2 = sol2(imag(sol2) == 0 & sol2 < -mu & sol2 >= (1 - mu)); % x + mu < 0 and x + mu - 1 >= 0
```

```
solution3 = sol3(imag(sol3) == 0 & sol3 >= -mu & sol3 < (1 - mu)); % x + mu >= 0 and x + mu
    - 1 < 0
solution4 = sol4(imag(sol4) == 0 & sol4 < -mu & sol4 < (1 - mu)); % x + mu < 0 and x + mu -
    1 < 0

solution = [solution1; solution2; solution3; solution4];

disp(vpa(solution, 6));
```

When dealing with equation with multiple solutions and absolute value function, it encountered several errors of displaying only few of the solutions they have. Therefore, I have divided each of the cases for absolution value to determine the sign and the interval of x. After the calculation, x values I've got is:

$$x_1 = 1.010035$$
$$x_2 = 0.990026$$
$$x_3 = -1.000001$$

The subscript of x doesn't represent the number of Lagrange point.

For **L2**,

$$x = 1.010035 \text{ AU}$$

For **L3**,

$$x = -1.000001 \text{ AU}$$

Besides, we also got a position for L1, $x = 0.990026$ AU. Therefore, we have successfully (and precisely) calculated and found the physical meaning of generalized Lagrange point.

Assignment 4

Stars and Universe
October 28, 2024
23-119 ONE CHOI

Problem 1

Using your own graphic tool, reproduce Figure 8-13 in the textbook by solving the Boltzmann and Saha equations for hydrogen. Use $P_e = 20\,\mathrm{N/m^2}$ for the electron pressure. The partition functions are $Z(\mathrm{H_{II}}) = 1$ and $Z(\mathrm{H_I}) = 2$. Find the excitation ratio (N_2/N_1) and the excited fraction (N_2/N) each for (a) Sirius with $T = 10,000\,\mathrm{K}$ and (b) the Sun with $T = 5,800\,\mathrm{K}$.

Solution

The entire process of deriving the equation is done on problem set. Therefore, I would only mention the results and important equations here.

Boltzmann distribution explains that the probability of the particle to have a certain energy level is proportional to the following:

$$P(E_i) \propto e^{-E_i/k_B T}$$

$Z(T)$ is a partition function. This concept emerged to calculate the total energy configuration, making the calculation for the probability for the particle to have certain energy level easy. The partition function is defined as:

$$Z(T) \equiv \sum_i g_i e^{-E_i/k_B T}$$

Here, g_i indicates i-th degeneracy factor, which multiplies redundancy. Using these concepts, Boltzmann's equation and Saha's equation were derived. Boltzmann's equation explains the distribution of how many particles are excited. The formula for Saha's equation is given as:

$$\frac{n_2}{n_1} = \frac{g_2}{g_1} e^{-(E_2 - E_1)/k_B T}$$

On the other hand, Saha's equation explains the distribution of ions on any state. The formula is given as:

$$\frac{n_{j+1} n_e}{n_j} = \frac{2Z_{j+1}(T)}{Z_j(T)} \left(\frac{2\pi m_e k_B T}{h^2} \right)^{3/2} e^{-\chi_{j,j+1}/k_B T}$$

This is the original form for Saha's equation, yet, when electron pressure is given (like in this problem), it is better to introduce it and transform. After the transformation, it looks like:

$$\frac{n_{j+1}}{n_j} = \frac{2k_B T Z_{j+1}(T)}{P_e Z_j(T)} \left(\frac{2\pi m_e k_B T}{h^2}\right)^{3/2} e^{-\chi_{j,j+1}/k_B T}$$

The reason why I have introduce these two formula is to derive the generalized equation for N_2/N_{tot}. Since there are only two ionization state for hydrogen, H_I and H_{II},

$$N_{\text{tot}} = N_I + N_{II}.$$

Also, since slightly excessive energy input can ionize the hydrogen, the number of excited states for neutral hydrogen can be approximated:

$$N_I \approx N_1 + N_2$$

Therefore,

$$\begin{aligned}
\frac{N_2}{N_{\text{tot}}} &= \frac{N_2}{N_I} \cdot \frac{N_I}{N_I + N_{II}} \\
&\approx \left(\frac{N_2}{N_1 + N_2}\right) \left(\frac{N_I}{N_I + N_{II}}\right) \\
&= \left(\frac{N_2/N_1}{1 + N_2/N_1}\right) \left(\frac{1}{1 + N_{II}/N_I}\right)
\end{aligned}$$

Using the approximations that problem provided, we are left with the generalized equation for this ratio. Note that degeneracy for the neutral hydrogen is expressed as: $g_i = 2i^2$.

$$\therefore \frac{N_2}{N_{\text{tot}}} = \left(\frac{4\exp(-1.184 \times 10^5/T)}{1 + 4\exp(-1.184 \times 10^5/T)}\right) \left(\frac{1}{1 + 0.001667 T^{5/2} \exp(-1.579 \times 10^5/T)}\right)$$

For complicated equation, MATLAB is known to provide accurate plot. Calculation for constants were done with more significant figures for the accuracy. The raw code for MATLAB is shown below:

```
T = linspace(5000, 25000, 500);

Ratio = (4 .* exp(-1.184e5 ./ T) ./ (1 + 4 .* exp(-1.184e5 ./ T))) ...
    .* (1 ./ (1 + 0.001666915 .* T.^(5/2) .* exp(-1.579e5 ./ T)));

figure;
set(gcf, 'Position', [100, 100, 700, 450]);
plot(T, Ratio, 'b-', 'LineWidth', 2);
ax = gca;
ax.FontSize = 12;
ylim([0 9.5e-6]);
xlabel('Temperature (T)', 'FontSize', 12, 'FontWeight', 'bold');
ylabel('N_2/N_{tot}', 'FontSize', 12, 'FontWeight', 'bold');
title('Plot of N_2/N_{tot} vs Temperature', 'FontSize', 14, 'FontWeight', 'bold');
grid on;
saveas(gcf, 'Balmer.pdf');
```

When the code above is rendered, the graph is saved as PDF file. The result is provided below:

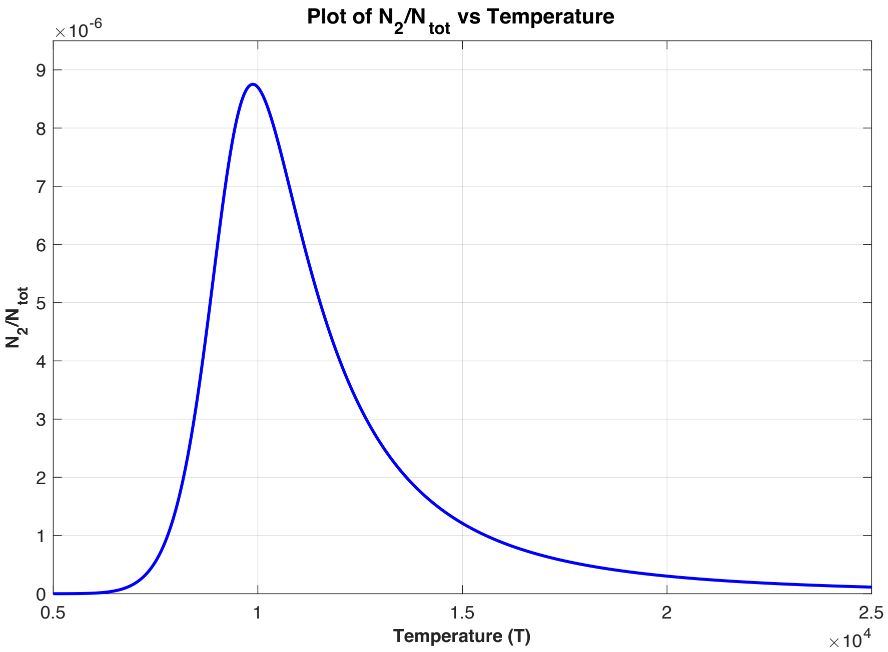


Figure 1: Balmer

Like our expectation that the ratio of excited hydrogen ($n = 2$) will be maximized at $T = 10,000$ K, this graph also showed the same tendency. Also, the value for the ratio is well calculated. Therefore, we can conclude that the graph is successfully determined.

The final step is to calculate the ratio of excited hydrogen ($n = 2$) for Sirius and Sun. To prevent any calculation errors, I have also used MATLAB for this calculation as well.

```
Ratio = @(T) (4 .* exp(-1.184e5 ./ T) ./ (1 + 4 .* exp(-1.184e5 ./ T))) ...
.* (1 ./ (1 + 0.001666915 .* T.^(5/2) .* exp(-1.579e5 ./ T)));

T1 = 5800;
T2 = 10000;

fprintf('Ratio at T = 5800: %.10e\n', Ratio(T1));
fprintf('Ratio at T = 10000: %.10e\n', Ratio(T2));
```

Results are organized below:

Table 1. Excited Hydrogen Atom Ratio by Stars

Star	**Temperature (T)**	**Ratio (N_2/N_{tot})**
Sirius	10,000 K	8.7020×10^{-6}
Sun	5,800 K	5.4508×10^{-9}

Like what we have expected, Sirius, the star that is $10,000$ K has higher ratio than the Sun, which is $5,800$ K.

Problem 2

(a) What is the energy of one photon of wavelength $\lambda = 300$ nm? Express your answer both in joules and in electron volts.

(b) An atom in the second excited state ($n = 3$) of hydrogen is just barely ionized when a photon strikes the atom. What is the wavelength of the photon if all energy is transferred to the atom?

Solution

(a) According to the quantum mechanics, energy of the photon is quantized. Planck has argued that:

$$E = h\nu$$

This is the amount of energy that one photon with the frequency of ν has. Using the relation that:

$$c = \lambda\nu$$

we obtain that:

$$E = \frac{hc}{\lambda}$$

Substituting the value from the problem,

$$E = 6.621 \times 10^{-19}\ \text{J}$$

If we convert this value into the unit of eV (electron volts), we obtain:

$$E = 6.621 \times 10^{-19}\ \text{J} \times \frac{1\ \text{eV}}{1.602177 \times 10^{-19}\ \text{J}} = 4.133\ \text{eV}$$

(b) Assuming no energy loss anywhere, we can write an energy conservation equation for this situation:

$$-R_y \cdot \frac{1}{3^2} + \frac{hc}{\lambda} = 0$$

Substituting in the values,

$$\lambda = \frac{9hc}{R_y} = 8.20 \times 10^{-7}\ \text{m} = 820\ \text{nm}$$

A wavelength of 820 nm (nanometers) falls in the near-infrared range of the electromagnetic spectrum.

Problem 3

To understand the relative importance of the different parameters in the Saha equation, perform the following experiment. Assume that $T = 5,000$ K, $N_e = 10^{15}$ cm^{-3}, and $\chi = 12$ eV. By what factor does the ionization ratio (N_+/N_0) change when we separately
(a) double the temperature
(b) double the electron density
(c) double the ionization potential

Which is more important during the temperature change, the expontential term or the $T^{3/2}$ term?

Solution

The purpose of this question is to determine which factor causes the entire value of the Saha's equation dominantly. Refer to the Saha's equation:

$$\frac{N_+}{N_0} = \frac{2Z_+(T)}{N_e Z_0(T)} \left(\frac{2\pi m_e k_B T}{h^2}\right)^{3/2} e^{-\chi/k_B T}$$

In the Saha's equation, there are many complex parameters to be determined, which seems like it would be very complicated to calculate every situation. However, the important thing to notice is that we are determining which factor is dominant. Let

$$f(x) = \left.\frac{N_+}{N_0}\right|_x$$

where x is the parameter that we are interested in. Then, we can compare the value of the $f(x)$ and $f(2x)$. And this can be done easily be setting $f(x) = 1$. Hence, we can see that the value of $f(2x)$ is then automatically a ratio of the value of $f(x)$.

(a) In this case, we are considering the changes in the temperature. let $f(T_0 = 5,000\text{ K}) = 1$. Then,

$$f(T) = \left(\frac{T}{5,000}\right)^{3/2}$$

If we plot this graph into 2D plane, we obtain

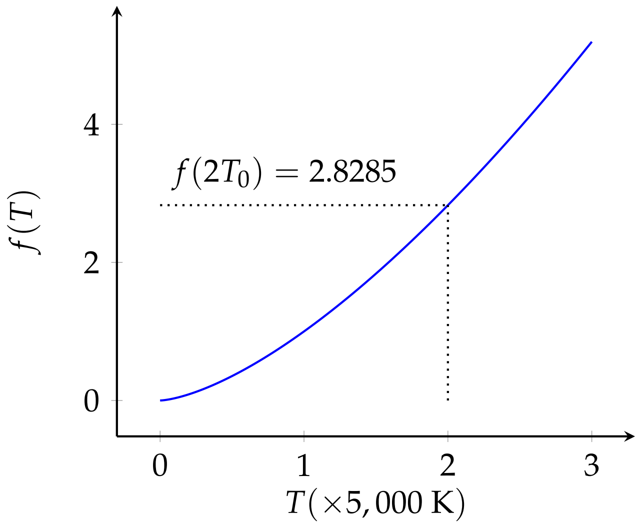


Figure 2: Ratio of Ions by Temperature

Therfore, we can notice that doubling the value of temperature increases the ratio by 2.8285 times.

(b) Here, we will consider the changes in electron density. From Saha's equation, we can see that the ratio is proportional to the inverse value of electron density. Thus, we can let:

$$f(N_e) = \frac{1}{N_e/10^{15}}$$

considering the initial value of the electron density. Note that function f changes in every situation and is temporary. Next step is to plot the graph.

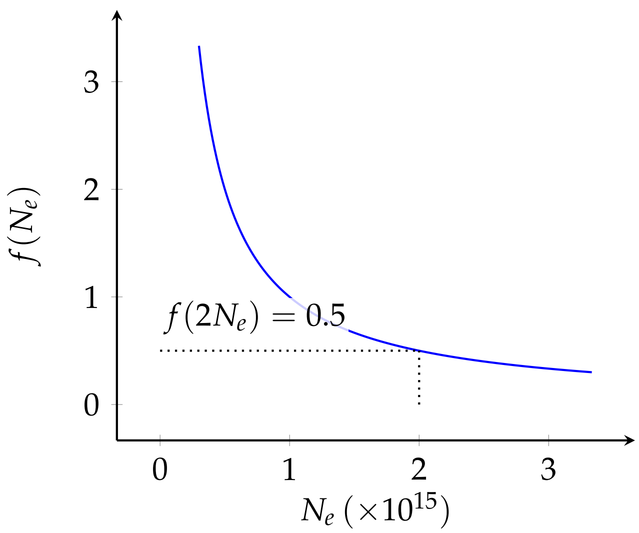


Figure 3: Ratio of Ions by Electrion Number Density

Hence, we can notice that doubling the value of electron density rather leads to decrease of total ratio into half.

(c) In this situation, we will consider the ionization potential, χ. Doing the same process, let:

$$f(\chi) = \exp\left(-\left(\frac{\chi}{12} - 1\right)\right)$$

The reason why this function looks more complicated is because it is an exponential function. Now, graphing this will give:

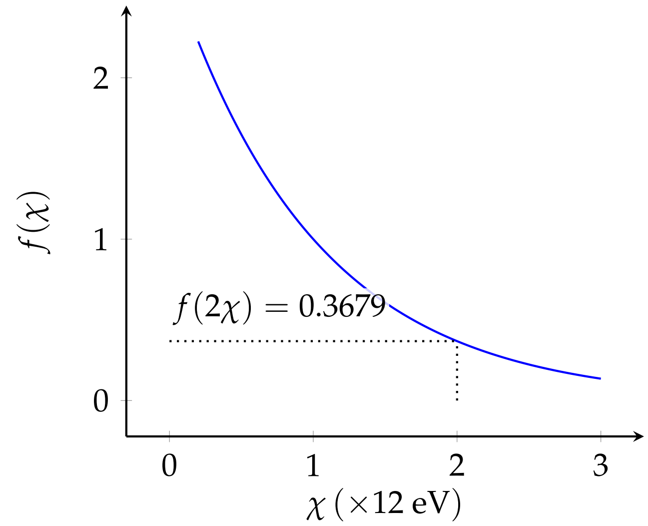


Figure 4: Ratio of Ions by Ionization Potential

Hence, we are left with the result that doubling the value of ionization potential will rather decrease the value of ratio by times of 0.3679.

Finally, let's examine which term is more important, the temperature term or the ionization potential term? In Saha's equation, two terms are multiplied. Assuming that as temperature doubles, ionization potential doubles as well, we can let a new function:

$$f(T, \chi) = \left(\frac{T}{5,000}\right)^{3/2} \cdot \exp\left(-\left(\frac{\chi}{12} - 1\right)\right)$$

Here, we do not have to plot 3D graph since we have assumed identical growth ratio for both parameters. Plotting this into 2D plane, we obtain:

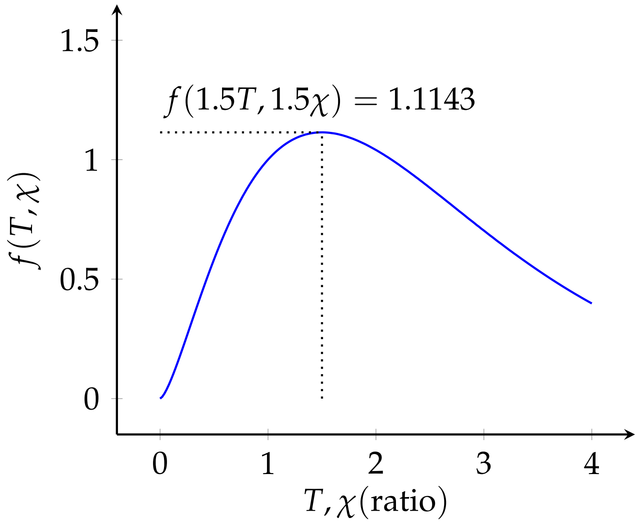


Figure 5: Ratio of Ions by Temperature and Ion Potential

We got interesting result here. Until the ratio of the temperature and the ionization potential is less than 1.5, the temperature term is dominant, increasing the value of the ratio. However, when it gets larger than 1.5, the exponential term gets more dominant, making the ratio smaller.

Problem 4

Estimate the line width from thermal Doppler broadening for the Ca II K line at 393.3 nm for a star at 3,000 K; a star at 6,000 K; and one at 12,000 K. Comment on the degree of importance of temperature on the Ca II K line broadening.

Solution

Thermal Doppler broadening is caused by a random motion of active particles. These particles then causes the Doppler effect, making the spectral line to get thicker. Using the root mean square value of the velocity, we can express the velocity by temperature:

$$\frac{1}{2}mv_{\text{rms}}^2 = \frac{3}{2}k_B T$$

At the same time, Doppler effect is expressed as:

$$\frac{\Delta\lambda}{\lambda_0} = \frac{v}{c}$$

Substituting velocity that we've determined by temperature above, we can expressed $\Delta\lambda$ with temperature.

$$\therefore \Delta\lambda = \frac{2\lambda_0}{c}\sqrt{\frac{3k_B T}{m}}$$

The reason why 2 is multiplied in the equation is because the random motion of the particles. Since they can move both back and forth, the Doppler effect can occur in both ways. The table below are the values determined after substituting constants.

Table 2. Thermal Doppler Broadening Width by Temperature

Temperature (T)	**Line Width ($\Delta\lambda$)**
3,000 K	3.585×10^{-12} m
6,000 K	5.070×10^{-12} m
12,000 K	7.170×10^{-12} m

Molar Mass for Ca^{2+}: 40.08 g/mol

Now, let's plot this graph to see how important the temperature is to the thermal Doppler broadening phenomenon.

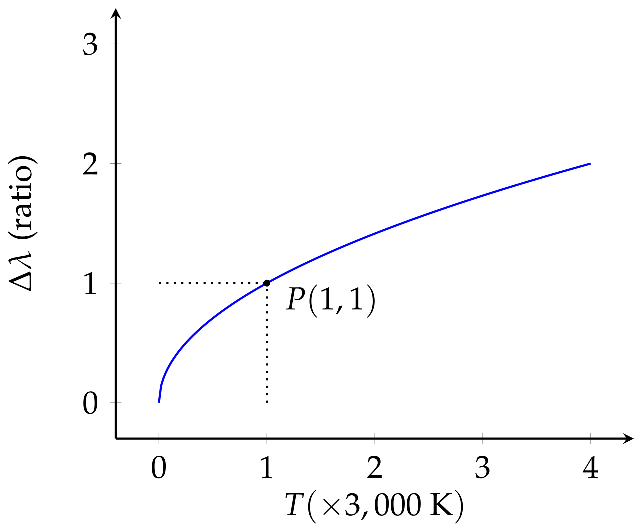


Figure 6: Ratio of Line Width by Temperature

This graph shows the ratio of $\Delta\lambda$, assuming that the ratio at $T = 3,000$ K is 1. As you can refer to the figure above, even though it is true that the ratio of $\Delta\lambda$ increases as the temperature rises, the speed of the increasing gradually slows down. This indicates that as temperature rises, it is not the temperature that is major factor. Yet, increasing the temperature to infinity will lead to infinitly thick spectral line broadening, theoretically.

Assignment 5

Stars and Universe
November 20, 2024
23-119 ONE CHOI

These are the solutions for problem 1, 4, 5, 14, 15, 16 of Chapter 13 in the textbook, Astronomy & Astrophysics (A&A).

Problem 1

The absorption spectra of four stars exhibit the following characteristics. What are the appropriate spectral types?

(a) The strongest features are titanium oxide bands.
(b) The strongest lines are those of ionized helium.
(c) The hydrogen Balmer lines are very strong, and some lines of ionized metals are present.
(d) There are moderately strong hydrogen lines, and lines of neutral and ionized metals are seen, but the Ca II, H, and K lines are the strongest in the spectrum.

Solution

The absorption spectra is well-organized in a single graph. The graph under shows absorption lines and the star's temperature.

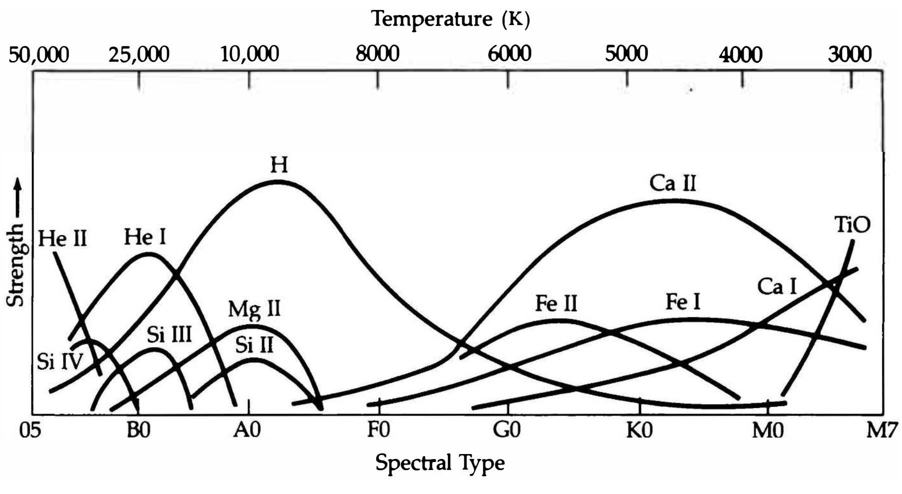


Figure 1: Absorption Lines and Temperature

(a) Titanium oxide bands, or TiO bands best represented in star with spectral type of M.

(b) Ionized helium is represented as He II. And this type of band is best represented on spectral type of O.

(c) The hydrogen Balmer lines are very strong across the spectral types of B, A, and F. Additional explanation says that some lines of ionized metals are present. Combining those two clues, spectral type of A best suits the condition.

(d) There are three clues illustrated here; hydrogen lines are moderately strong, lines of metals that are neutral and ionized are seen, and Ca II, H, K lines are the strongest. These conditions are best represented on spectral type of F.

Problem 4

Which parameter in the Saha ionization-equilibrium equation is most important in explaining the spectral differences between

(a) giants and dwarfs of spectral type G
(b) B and A dwarfs

Solution

Recall the Saha's equation from Chapter 8.

$$\frac{n_{i+1}}{n_i} = \frac{2Z_{i+1}k_B T}{P_e Z_i} \left(\frac{2\pi m_e k_B T}{h^2}\right)^{3/2} e^{-\chi/k_B T}$$

We want to highlight the relation of the parameters and the ratio of ions.

$$\frac{N_{i+1}}{N_i} \propto \left(\frac{(k_B T)^{3/2}}{N_e}\right) \exp(-\chi_i/k_B T)$$

Here, χ_i is the ionization potential from stage i to stage $i+1$. We can apply the log scale to clearly distinguish the role of parameters.

$$\log\left(\frac{N_{i+i}}{N_i}\right) = \frac{5}{2}\log T - \frac{5040}{T}\chi_i - \log P_e + C$$

C represents the constants that appear due to application of the logarithmic scale. By utilizing this equation, we can distinguish which parameter is important in each situations.

(a) For giants and dwarfs of the same spectral type, the only difference is the radius, while the temperature stays constant. When the radius changes, the electron pressure changes. Therefore, P_e is the important factor to consider in this situation.

(b) For two stars with same size but different spectral type, their temperature are different. Therefore, the electron pressure is now constant, and temperature is the now the important factor.

Problem 5

(a) What is the ratio of the surface gravities of a K0 V star and a K0 I star?
(b) If both stars had the same atmospheric temperatures and opacities, what would be the approximate ratio of strengths of the Ca II K lines in their spectra? (The ionization potential of calcium is 6.1 eV .)
(c) What assumptions did you make to answer part (b)?
(d) What is the ratio of the mean densities of these two stars?
(e) If the atmospheric electron densities of these stars were directly proportional to their mean densities, would your answer to (b) remain unchanged? Why?

Solution

(a) We will use the Pogson's equation:

$$M_1 - M_2 = -2.5 \log \left(\frac{L_1}{L_2} \right)$$

where M_1 is an absolute magnitude of a star K0 V, and M_2 is an absolute magnitude of a star K0 I. These magnitudes can be determined by plotting in the Hertzsprung-Russell diagram. Only considering the proportionality of the equation,

$$L = C \cdot 10^{-0.4M}, \text{ C is constant}$$

Since the two stars have the same spectral type, the temperature is identical, thus:

$$\frac{R_1}{R_2} = \sqrt{\frac{L_1}{L_2}} = \sqrt{10^{-0.2(M_1 - M_2)}}$$

The ratio of the gravity is equivalent to the ratio of the gravitational acceleration constant.

$$\frac{g_1}{g_2} = \frac{m_1 R_2^2}{m_2 R_1^2}$$

Let the ratio of two stars be κ. Then,

$$\frac{g_1}{g_2} = 10^{-0.2(M_1 - M_2)} \cdot \kappa$$

(b) Since they have the exact same atmospheric temperatures and opacities, due to figure 1, they must have the same spectral line intensity ratio.

According to Boltzmann's equation,

$$\frac{n_2}{n_1} = \frac{g_2}{g_1} e^{-(E_2-E_1)/k_B T}$$

If we substitute proper values for the equation, we will get an approximate intensity for spectral lines. For instance,

$$\frac{6.1 \text{ eV}}{k_B T} \approx 12.25$$

(c) Assumptions:

- Same atmospheric composition
- No external impact
- Hydrostatic equilibrium
- Thermal equilibrium
- Myriad numbers of atoms in the atmosphere

(d) Let κ (mass ratio), M_1, and M_2 be defined. From (a), we are aware that

$$\frac{m_1 R_2^2}{m_2 R_1^2} = 10^{-0.2(M_1-M_2)} \cdot \kappa$$

Here,

$$\frac{\rho_1}{\rho_2} = \frac{M_1}{M_2} \cdot \left(\frac{R_2}{R_1}\right)^3 = \kappa \left(10^{-0.2(M_1-M_2)}\right)^{3/2}$$

(e) If the atmostpheric electron densities of the stars changes with different conditions, figure 1 would not be the same as how it was. Instead, according to that, the graphs will shift or stretch in either directions. Even though that since both stars have the same temperature, the intensity of the spectral lines are the same, the value will change afterwards.

This phenomenon happens due to Saha's equation. When calculating the ratio of ions to total numbers of atoms, Saha's equation must be used, and Electron pressure is highly affecting the Saha's equation.

In addition to this, if electron density has changed, it also affects the degeneracy factor of the electron, g_e, to change. Therefore, if a mean density is proportional to the electron density, the answers for (b) will change.

Problem 14

Estimate the distances to the following clusters from their color magnitude figures (use Table A4-3 to convert $B - V$ to spectral type):

(a) the Pleiades (Figure 13-9)
(b) M3 (Figure 13-10)

Solution

Before the analysis, here is the table A4-3 from the textbook.

TABLE A4–3 Stellar Characteristics by Spectral Type and Luminosity Class

Spectral Type	M_v			$B-V$			T_{eff} (K)			BC	$R/R_\odot$			$M/M_\odot$		
	V	III	Ib*	V	III	I	V	III	I	V	V	III	I	V	III	I
O5	−6.0			−0.32	−0.32	−0.32	50,000			−4.30	18			40		100
B0	−4.1	−5.0	−6.2	−0.30	−0.30	−0.24	27,000			−3.17	7.6	16	20	17		50
B5	−1.1	−2.2	−5.7	−0.16	−0.16	−0.09	16,000			−1.39	4.0	10	32	7		25
A0	+0.6	−0.6	−4.9	0.00	0.00	+0.01	10,400			−0.40	2.6	6.3	40	3.6		16
A5	+2.1	+0.3	−4.5	+0.15	+0.15	+0.07	8200			−0.15	1.8		50	2.2		13
F0	+2.6	+0.6	−4.5	+0.30	+0.30	+0.24	7200			−0.08	1.3		63	1.8		13
F5	+3.4	+0.7	−4.5	+0.45	+0.45	+0.45	6700	6500	6200	−0.04	1.2	4.0	80	1.4		10
G0	+4.4	+0.6	−4.5	+0.60	+0.65	+0.76	6000	5500	5050	−0.06	1.04	6.3	100	1.1	2.5	10
G5	+5.2	+0.3	−4.5	+0.65	+0.86	+1.06	5500	4800	4500	−0.10	0.93	10	126	0.9	3	13
K0	+5.9	+0.2	−4.5	+0.81	+1.01	+1.42	5100	4400	4100	−0.19	0.85	16	200	0.8	4	13
K5	+8.0	−0.3	−4.5	+1.18	+1.52	+1.71	4300	3700	3500	−0.71	0.74	25	400	0.7	5	16
M0	+9.2	−0.4	−4.5	+1.39	+1.65	+1.94	3700	3500	3300	−1.20	0.63		500	0.5	6	16
M5	+12.3	−0.5	−4.5	+1.69	+1.85	+2.15	3000	2700		−2.10	0.32			0.2		

*All class Ia stars have an absolute visual magnitude of −0.7.

BC is bolometric correction.

Figure 2: Table A4-3

(a) Here is a figure for Pleiades color=magnitude diagram.

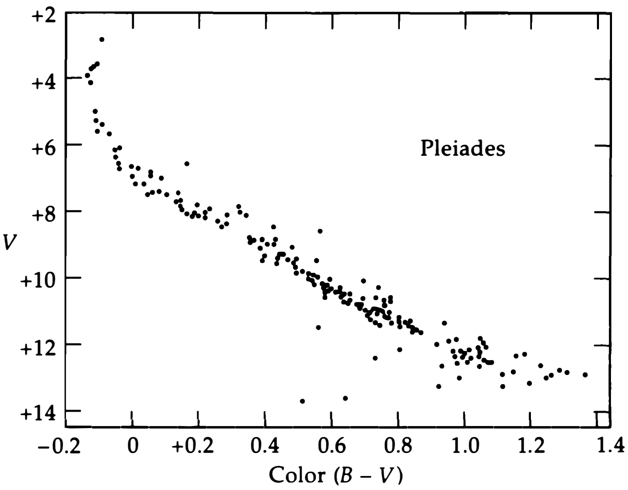


Figure 3: Color=magnitude diagram of Pleiades

The method of doing this process if the use the data from the table, and insert the M_V value that fit with $B - V$ value. Since the figure only shows the apparent magnitude of the stars, we need to find absolute magnitude for corresponding $B - V$. To simplify the entire main

sequence fitting process, I have simply investigated for the two end points to determine the distance between the lines.
The horizontal distance between the two linear line is about 5.2 $(= m - M)$
According to the Pogson's equation, we have

$$m - M = 5 \log d - 5$$

If we transform the equation so that we can derive the distance,

$$d = 10^{\left(\frac{m-M+5}{5}\right)}$$

The experimentally measured distance for Pleiades cluster is

$$d_{\text{p,exp}} = 109.65 \text{ parsec}$$

The theoretical value for the distance to Pleiades is: $d_p = 136.2$ parsec. Although it shows some significant discrepency, I presume that tendency was well shown and the process was done without any errors.

(b) We have to apply the same method to this cluster. First of all, here is the figure of color-magnitude diagram.

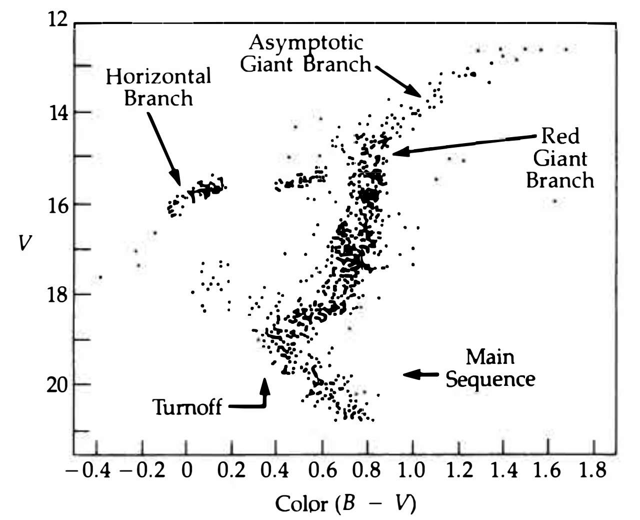


Figure 4: Color-magnitude diagram for M3 cluster

Apply the same Pogson's equation for this graph, we are yielded with that the distance between the two parallel lines is 15.2. Thus $m - M = 15.2$.

One thing noticable is that, as it could be expected, the distance to M3 is extremely far away. Moreover, the actual distance to M3 cluster is $10,000$ parsec. To compare with the value that we've got, it is quite surprising that the data were very accurate and well-calculated.

Problem 15

Estimate the radii of both a main-sequence M star (M V) and a red supergiant (M I) using information from the H-R diagrams in the text.

Solution

Here is an image of Hertzsprung-Russell diagram from the textbook.

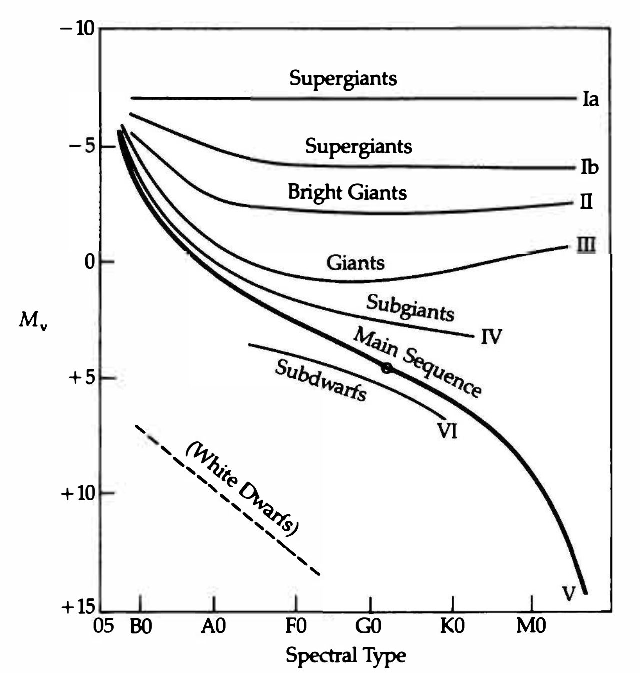


Figure 5: Hertzsprung-Russell diagram

Let's consider the star with spectral type especially with M0. We can utilize the table that is used at the previous question. According to Table A4-3, for main-sequence star, the absolute magnitude M_v is around +9.2. And for the red supergiant, the absolute magnitude is around -4.5.

$$M_1 = +9.2 \text{ and } M_2 = -4.5$$

Temperature of each of the stars are also revealed in the table. $T_1 = 3700$ K and $T_2 = 3300$ K. For radius, it is given as $R_1 = 0.63R_\odot$ and $R_2 = 500R_\odot$ on the table, but let's say we do not know this value, and compare with the estimated value. For the sun, the absolute magnitude is known as +4.83 and the temperature is 5,778 K, and let this be given.

$$\frac{M_1}{M_\odot} = 1.905 = -2.5\log\left(\frac{L_1}{L_\odot}\right) = -5\log\left(\frac{R_1 T_1^2}{R_\odot T_\odot^2}\right)$$

$$\therefore R_1 = \frac{T_\odot^2}{T_1^2} \cdot 10^{-M_1/5M_\odot} \cdot R_\odot$$

And this applies for R_2 as well.

$$R_2 = \frac{T_\odot^2}{T_2^2} \cdot 10^{-M_2/5M_\odot} \cdot R_\odot$$

Substituting the values into the equation we have derived,

$$R_{1,\text{exp}} = 1.014R_\odot \text{ and } R_{2,\text{exp}} = 4.708R_\odot$$

The error seems to be extremely huge for both of the cases. Even though the equations were established properly, there are various physical phenomena to take into account when calculating the radius.

Problem 16

Using the H-R diagram in Figure 13-7 and the relationship between temperature and spectral type in Figure 13-6, estimate how many times larger is Betelgeuse than

(a) Antares
(b) β Crucis
(c) α Centauri

Solution

Refer to the figure from the textbook.

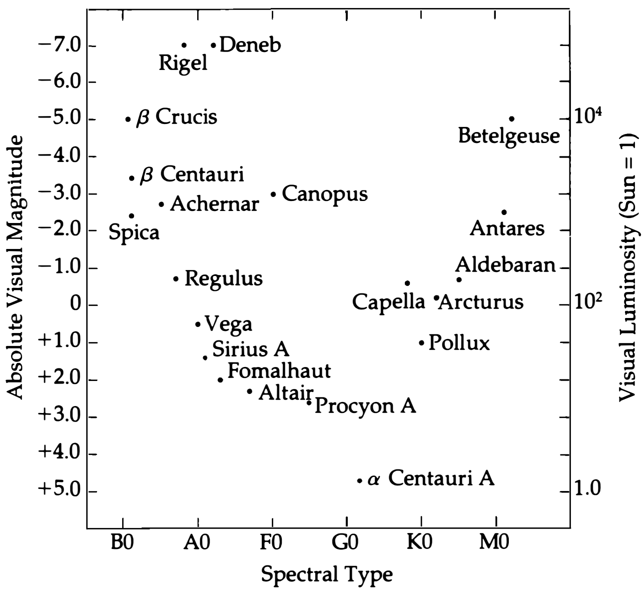


Figure 6: Hertzsprung-Russell diagram for the brightest stars in the sky

(a) The luminosity is expressed as:

$$L = 4\pi R^2 \sigma T^4$$

according to Stefan-Boltzmann's law. For Antares, it can be assumes that it has the same spectral type with the Betelgeuse, thus they have about the same temperature.

$$\frac{L_B}{L_A} = \frac{R_B^2}{R_A^2}$$

Antares looks like it is resting on the center of the 10^2 and 10^4, where axis is log scaled. Hence, $L_B/L_A = 10$.

$$\therefore R_B \approx 3.16 R_A$$

Here, $\sqrt{10} \approx 3.1623$.

(b) β Crucis and the Betelgeuse has the same luminosity. we can utilize this to calculate the radius ratio. According to the graph, β Crucis has spectral type of B1, and B1 star has the temperature of about 23,000 K. The Betelgeuse has the spectral type of about M1, and M1 star has the temperature of about 3,700 K. Since they have the same luminosity,

$$\frac{R_B}{R_\beta} = \frac{T_\beta^2}{T_B^2} = \frac{23,000^2}{3,700^2}$$

$$\therefore R_B \approx 38.64 R_\beta$$

(c) Both temperature and the luminosity is different for the α Centauri. In this case, both of the parameters should be taken into the account.

$$\frac{R_B}{R_\alpha} = \frac{\sqrt{L_B} T_\alpha^2}{\sqrt{L_\alpha} T_B^2} = \sqrt{10^{-0.4(M_\alpha - M_B)}} \cdot \frac{T_\alpha^2}{T_B^2}$$

According to figure 6, α Centauri seems to have a spectral type of G2, with absolute magnitude of +5.3. Therefore, the temperature is 5,800 K, similar to our sun.

$$\therefore R_B = \frac{\sqrt{L_B} T_\alpha^2}{\sqrt{L_\alpha} T_B^2} \cdot R_\alpha \approx 0.0214 R_\alpha$$

Assignment 6

Stars and Universe
December 6, 2024
23-119 ONE CHOI

These are the solutions for problem 14 and 15 of Chapter 15 and for problem 5, 10, 11, and 12 for Chapter 16 in the textbook, Astronomy & Astrophysics (A&A).

Problem 15 - 14

Consider an extended region filled with neutral hydrogen gas at a density of nm_H and temperature T, where m_H is the mass of the hydrogen atom. Now consider a small spherical volume of radius R inside this region where the density is slightly higher than in the surrounding region.

(a) Show that the radius of this volume must satisfy the condition $R \geq \left(3k_BT/2\pi Gnm_H^2\right)^{1/2}$ in order for gravitational collapse of the denser region to occur. (Hint: Obtain an expression for gravitational pressure by integrating Equation 16-1 over r.)
(b) Obtain a lower limit for the mass of the collapsing volume.
(c) For the general interstellar medium, we can take the approximate values $n = 10^6$ /m^3 and $T = 100$ K. Obtain a numerical value for the mass limit in solar masses.
(d) Is a collapsing region of the general interstellar medium likely to produce a single star like the Sun?

Solution

(a) According to the hydrostatic equilibrium equation,

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM(r)}{r^2}\rho(r)$$

For the approximation given in the problem, $\rho = nm_H$ (constant).

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM(r)}{r^2}nm_H$$

As indicated in the equation, the mass changes by the distance from the center. Since we have assumed a constant density,

$$M(r) = \left(\frac{r}{R}\right)^3 M = \left(\frac{r}{R}\right)^3 \frac{4}{3}\pi R^3\rho = \frac{4}{3}\pi nm_H r^3$$

Substituting this into hydrostatic equilibrium equation, at the center of the star, the pressure due gravitational force is expressed as:

$$\int \mathrm{d}P = \int -\frac{4}{3}\pi G n^2 m_H^2 r \, \mathrm{d}r$$

$$\begin{aligned}
\therefore P_{\text{grav}} &= \int_R^0 -\frac{4}{3}\pi G n^2 m_H^2 r \, \mathrm{d}r \\
&= -\frac{2}{3}\pi G n^2 m_H^2 r^2 \Big|_R^0 \\
&= \frac{2}{3}\pi G n^2 m_H^2 R^2
\end{aligned}$$

The region will collapse gravitationally when $P_{\text{grav}} > P_{\text{thermal}}$. Setting $P_{\text{grav}} = P_{\text{thermal}}$, we get:

$$\frac{2}{3}\pi G n^2 m_H^2 R^2 = n k_B T.$$

Simplifying the entire equation in respect to R, then the problem is solved.

$$R \geq \left(\frac{3k_B T}{2\pi G n m_H^2}\right)^{1/2}.$$

(b) Continue using the information that we've acquired through problem (a), we can substitute a critical radius into a mass equation.

$$M = \frac{4}{3}\pi R^3 \rho = \frac{4}{3}\pi R^3 n m_H$$

$$M = \frac{4}{3}\pi \left(\frac{3k_B T}{2\pi G n m_H^2}\right)^{3/2} n m_H.$$

(c) The numerical values were provided on the question. After checking if all the unit of the numerical values were provided in SI unit, we can substitute those values.

$$M = \frac{4}{3}\pi \left(\frac{3 \cdot 1.38 \times 10^{-23} \cdot 100}{2\pi \cdot 6.674 \times 10^{-11} \cdot 10^6 \cdot (1.67 \times 10^{-27})^2}\right)^{3/2} \cdot 10^6 \cdot 1.67 \times 10^{-27}$$

In conclusion,

$$M = 4.6592 \times 10^{34}\ \text{kg} = 2.3424 \times 10^5 M_\odot$$

(d) The mass of the interstellar medium is calculated to much larger than the mass of the sun. If the mass is excessively large, the state is unstabled, thus it may rather breakdown into several stars and possibly forming a star cluster.

Problem 15 - 15

A star count is made on a photograph that contains a dark nebula. Ten times fewer stars are counted in front of the cloud than are found on a region away from the cloud with the same solid angle. For ease of calculation, assume that the dark cloud is completely opaque, that the stars are uniformly distributed in space, that all stars have absolute magnitudes of 5.0, and that the limiting magnitude of the photograph is 15.0. Calculate an approximate distance to the cloud.

Solution

Given data says that the dark cloud reduces the star count by a factor of 10. The absolute magnitude of stars is $M = 5.0$, limiting apparent magnitude of the photograph is $m = 15.0$, the stars are uniformly distributed in space, and the dark cloud is assumed to be completely opaque (blocks all background stars).

First, let's assume that there is no dark clouds blocking the starlight. Pogson's equation says that

$$m - M = 5\log(d_{\text{max}}) - 5$$

Rearranging for d_{max},

$$d_{\text{max}} = 10^{\frac{m-M+5}{5}}.$$

$$\therefore d_{\text{max}} = 10^{\frac{15-5+5}{5}} = 10^3 = 1000 \text{ pc}.$$

So, stars are visible up to a distance of 1000 pc if there is no dark cloud. Now assume that the dark clouds are back. The dark nebula blocks all stars behind it. Since only 10% of the stars are visible, the stars in front of the cloud must occupy 10% of the visible volume. Thus,

$$\frac{d^3_{\text{cloud}}}{d^3_{\text{max}}} = 0.1.$$

Substituting and calculating for the value:

$$d_{\text{cloud}} = 10^{8/3} = 10^{2.67} \approx 464.16 \text{ pc}.$$

Problem 16 - 5

Although detailed models of stellar structure require the use of complex computer codes, simple scalings can be obtained by making rough approximations. For a variable x, we can substitute $\Delta x/\Delta r$ for dx/dr in order to obtain a crude result. (This method is a rough version of a numerical technique called finite differences.)

(a) Use the equation of hydrostatic equilibrium (161) to show that the central pressure scales as $P_c \propto M^2/R^4$. Substitute $\Delta P/\Delta r$ for $\mathrm{d}P/\mathrm{d}r$ and take this difference between $r = 0$ and $r = R/2$, that is, $\Delta P/\Delta r \approx \dfrac{P(r = R/2) - P_c}{(R/2 - 0)}$. You may assume that $P(r = R/2)$ is negligible

compared with P_c. Also, substitute the mean density of the star $\langle\rho\rangle$ for $\rho(r)$.
(b) Now use the radiative transport equation and the same method as in part (a) to approximate $\mathrm{d}T/\mathrm{d}r$ and obtain the theoretical mass-luminosity relationship, $L \propto M^3$. Assume that $\kappa(r)$ is constant and that $T(r) \propto T_c$.
(c) Use the same method to obtain a rough relationship between the effective temperature and mass of a main-sequence star. (No other variables should appear in the proportionality.)
(d) Combine your answers to parts (b) and (c) to obtain a relationship between T and L. Use the H-R diagrams (Figures 137 and 139) and Table A4-3 to compare the temperature of a star of $L = 10L_\odot$ with that of the Sun. How does this compare with your theoretical T-L relationship (be quantitative)?

Solution

(a) The simplest assumptions that we can make is that the gradient of the pressure over the distance is linear, which make the differential equation extremely easy to solve. The question requires us to find the proportionaity relation, thus unimportant constants were most likely removed during the process.

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM}{r^2}\rho$$

Assuming that the density is constant,

$$\rho = \frac{M}{\frac{4}{3}\pi R^3}$$

Therefore, substituting the value gives

$$\frac{\mathrm{d}P}{\mathrm{d}r} \propto \frac{M^2}{R^5}$$

$$\therefore \Delta P \propto \int \frac{M^2}{R^5}\mathrm{d}r \propto \frac{M^2}{R^4}$$

An assumption successfully leads to the result that should be derived.

(b) From the previous questions, hydrostatic equilibrium equation and ideal gas state equation was mentioned, using these,

$$P \propto \frac{M\rho}{R} \propto \rho T$$

$$\therefore T \propto \frac{M}{R}$$

The energy flux due to radiation in the stellar interior is written by:

$$L \propto \frac{RT_c^4}{\kappa\rho},$$

Substituting $\rho \propto M/R^3$ and assuming that the opacity of the star is constant throughout the radial distance,

$$\therefore L \propto M^3.$$

(c) According to Stefan-Boltzmann law of radiation, $L = 4\pi R^2 \sigma T^4$, where T is the effective temperature of the star. Since,

$$R \propto M^{1/3},$$

thus,

$$L \propto M^{2/3} T^4$$

Somewhat exact result should be

$$T^4 \propto M^{7/3},$$

yet roughly saying, it is can be expressed as following:

$$T \propto M^{1/2}.$$

(d) Writing down relevant relations,

$$L \propto M^{8/3} \propto T^{16/3} \approx T^5$$

Now, let's compare this result with the observatory results. For the sun and the star with $L = 10L_\odot$,

Table 1

Value Type	Observed Value (Altair)	Theoretical Results	Percent Difference
Luminosity (L)	10.6 $L_\odot$	10.6 $L_\odot$ (assume true)	
Mass (M)	1.79 $M_\odot$	2.424 $M_\odot$	15.05 %
Temperature (T)	7,900 K	9,265 K	7.952 %

Observed Values were from Altair star

These values were based on Table A4-3; it is provided below. The value of luminosity is not explicitly mentioned, thus I have used the known value of Altair, which had luminosity of around 10.

TABLE A4–3 Stellar Characteristics by Spectral Type and Luminosity Class

Spectral Type	M_v			$B-V$			T_{eff} (K)			BC	$R/R_\odot$			$M/M_\odot$		
	V	III	Ib*	V	III	I	V	III	I	V	V	III	I	V	III	I
O5	−6.0			−0.32	−0.32	−0.32	50,000			−4.30	18			40		100
B0	−4.1	−5.0	−6.2	−0.30	−0.30	−0.24	27,000			−3.17	7.6	16	20	17		50
B5	−1.1	−2.2	−5.7	−0.16	−0.16	−0.09	16,000			−1.39	4.0	10	32	7		25
A0	+0.6	−0.6	−4.9	0.00	0.00	+0.01	10,400			−0.40	2.6	6.3	40	3.6		16
A5	+2.1	+0.3	−4.5	+0.15	+0.15	+0.07	8200			−0.15	1.8		50	2.2		13
F0	+2.6	+0.6	−4.5	+0.30	+0.30	+0.24	7200			−0.08	1.3		63	1.8		13
F5	+3.4	+0.7	−4.5	+0.45	+0.45	+0.45	6700	6500	6200	−0.04	1.2	4.0	80	1.4		10
G0	+4.4	+0.6	−4.5	+0.60	+0.65	+0.76	6000	5500	5050	−0.06	1.04	6.3	100	1.1	2.5	10
G5	+5.2	+0.3	−4.5	+0.65	+0.86	+1.06	5500	4800	4500	−0.10	0.93	10	126	0.9	3	13
K0	+5.9	+0.2	−4.5	+0.81	+1.01	+1.42	5100	4400	4100	−0.19	0.85	16	200	0.8	4	13
K5	+8.0	−0.3	−4.5	+1.18	+1.52	+1.71	4300	3700	3500	−0.71	0.74	25	400	0.7	5	16
M0	+9.2	−0.4	−4.5	+1.39	+1.65	+1.94	3700	3500	3300	−1.20	0.63		500	0.5	6	16
M5	+12.3	−0.5	−4.5	+1.69	+1.85	+2.15	3000	2700		−2.10	0.32			0.2		

*All class Ia stars have an absolute visual magnitude of −0.7.

BC is bolometric correction.

The percentage difference of the theoretical value and the observed values all fall within 10 % interval. It implies that the numerical analysis and derivation was somewhat appropriate and accurate, yet there were still discrepencies occurring.

Complex factors were involved in the error. To list a few, these are potential error causing factors. First is too much approximations. Approximation makes the calculation easier and convenient, yet, in this calculation, there were too much approximations such as constant density, roughing out process, and constant opacity, etc. Secondly, the stellar model is too simple to analyze the accurate value. The star is extremely complicated and sensitive in some ways. Model provides great insight to intuitively understand how it works, but it is not the best. Finally, the every stars are different. These models explain the star's mechanism in very broad way, but like every people are different, every stars are different, too. These simple equations are insufficient to compare all the stars with 10 times the luminosity of the sun.

Problem 16 - 10

Estimate the energy available and the lifetime for the helium-burning phase in a $1M_\odot$ star:

(Note: The weight of ^{4}He is 4.0026, and the weight of ^{12}C is 12.0000.)
(b) What fraction of the available mass of 3 helium nuclei is liberated in the form of energy in the triple-alpha reaction? Compare this to the fraction of available mass liberated in the protonproton reaction.
(c) Assume that approximately 10% of the original mass of the star is in the form of ^{4}He in the stellar core during the helium-burning phase. Estimate the total energy available from the triple-alpha process.
(d) During the helium-core-burning phase, some hydrogen burning is also occurring in a shell. Thus the star's luminosity is not due only to helium burning. Keeping this in mind, assume that the typical luminosity from helium burning is $10^2 L_\odot$. Estimate the lifetime of the helium-core-burning phase.

Solution

(a) In a triple-alpha process of nuclear fusion, the mass of hydrogens were lost and converted into the energy. The masses are: $m(^4\text{He}) = 4.0026\,\text{u}$ and $m(^{12}\text{C}) = 12.0000\,\text{u}$. The mass difference is:

$$\Delta m = 3 \times m(^4\text{He}) - m(^{12}\text{C}) = 3 \times 4.0026 - 12.0000 = 0.0078\,\text{u}.$$

To convert the mass into the energy, we need to use the Einstein's mass-energy equivalence equation, $E = mc^2$. Energy for 1 u is:

$$E = 1\,\text{u} \cdot c^2 = \frac{0.001\ \text{kg}}{6.626 \times 10^{23}} \cdot \left(2.9979 \times 10^8\ \text{m/s}\right)^2 = 1.4924 \times 10^{-10}\ \text{J}$$

Using this equation for conversion,

$$E = 0.0078\,\mathrm{u}\cdot c^2 = 0.0078 \times 1.4924 \times 10^{-10}\,\mathrm{J} = 1.1641 \times 10^{-12}\,\mathrm{J}$$

(b) The question is straightforward,

$$\text{Fraction} = \frac{0.0078}{3 \times 4.0026} = 0.00065 = 0.065\%$$

(c) Assuming that 10% of star's mass is neutral helium, initially, and all of those masses can be converted into energy, The fraction of the mass used for the energy source is preserved, therefore,

$$E = 0.1 \times 0.00065 \times 1M_\odot \times \left(2.9979 \times 10^8\,\mathrm{m/s}\right) = 1.1619 \times 10^{43}\,\mathrm{J},$$

which is a tremendous amount of energy released.

(d) During the helium-core-burning phase, the typical luminosity due to helium burning is $10^2 L_\odot$, where $L_\odot = 3.846 \times 10^{26}\,\mathrm{W}$. The luminosity has a meaning of energy released per time. Therefore the life span is the total energy released from the star divided by the luminosity. This method is approximated, yet it provides the information about the star's lifetime with accurate scale.

$$\tau = \frac{E_{\text{total}}}{L} = \frac{1.16 \times 10^{44}}{3.846 \times 10^{28}} \approx 3.02 \times 10^{15}\,\mathrm{s}.$$

Converting the time τ into year unit,

$$\tau = 9.558 \times 10^6\,\text{years}.$$

Problem 16 - 11

11. Consider a model star in which the density $\rho(r)$ goes as ρ_0 in the core $(r < r_0)$, $\rho_0(r_0/r)^2$ in the region between the core and surface $(r_0 < r < R)$, and 0 outside the surface $(r > R)$.

(a) Find an expression for $M(r)$.
(b) If the stars mass is $1M_\odot$ at $R = R_\odot$, and $r_0 = 0.1R$, what is the value of ρ_0?
(c) Find an expression for $P(r)$.

Solution

(a) The function for the density is symmetric in respect to radial distance. It allows easier integration for the function. The density function is given as:

$$\rho(r) = \begin{cases} \rho_0 & \text{for } r < r_0, \\ \rho_0 \left(\frac{r_0}{r}\right)^2 & \text{for } r_0 \le r \le R, \\ 0 & \text{for } r > R. \end{cases}$$

Since the density differs as the radial distance change, the mass of the star will differ by the distance.

$$\mathrm{d}M = \rho \mathrm{d}V = \rho \cdot 4\pi r^2 \mathrm{d}r$$

for $r < r_0$,

$$\int \mathrm{d}M = \int_0^r \rho_0 \cdot 4\pi r^2 \mathrm{d}r = \frac{4}{3}\pi r^3 \rho_0$$

for $r_0 < r$,

$$M = \frac{4}{3}\pi r_0^3 \rho_0 + \int_{r_0}^{r} \rho_0 \left(\frac{r_0}{r}\right)^2 \cdot 4\pi r^2 \mathrm{d}r = \frac{4}{3}\pi r_0^3 \rho_0 + 4\pi \rho_0 r_0^2 (r - r_0)$$

To conclude,

$$\therefore M(r) = \begin{cases} \frac{4}{3}\pi r^3 \rho_0 & \text{for } r < r_0, \\ \frac{4}{3}\pi r_0^3 \rho_0 + 4\pi \rho_0 r_0^2 (r - r_0) & \text{for } r_0 \leq r \leq R. \end{cases}$$

(b) Inserting the values that were given in the problem, it leads to following:

$$\begin{aligned} M &= \frac{4}{3}\pi \left(\frac{R}{10}\right)^3 \rho_0 + 4\pi\rho \left(\frac{R}{10}\right)^2 \left(R - \frac{R}{10}\right) \\ &= \frac{4}{3}\pi\rho_0 \left(\frac{R}{10}\right)^3 \cdot 28 \\ &= \frac{112}{3000}\pi\rho_0 R^3 \end{aligned}$$

$$\therefore \rho_0 = \frac{3000M}{112\pi R^3}$$

(c) The same process can be applied here as well. Since the pressure is applied from outwards, it should be integrated from the outer shell. For $r > r_0$,

$$\begin{aligned} P &= \int \mathrm{d}P \\ &= \int_R^r -\frac{GM(r)}{r^2}\rho(r)\mathrm{d}r \\ &= \int_R^r -\left(-\frac{8}{3}\pi r_0^3 \rho_0^2 + 4\pi \rho_0^2 r_0^2 r\right) \frac{G}{r^4}\mathrm{d}r \\ &= \int_R^r \frac{8}{3}\pi r_0^3 \rho_0^2 \frac{G}{r^4}\mathrm{d}r - \int_R^r 4\pi\rho_0^2 r_0^2 \frac{G}{r^3}\mathrm{d}r \\ &= -\frac{8}{9}\pi r_0^3 \rho_0^2 \frac{G}{r^3}\bigg|_R^r + 2\pi \rho_0^2 r_0^2 G \frac{1}{r^2}\bigg|_R^r \\ &= \frac{8}{9}\pi G r_0^3 \rho_0^2 \left(\frac{1}{R^3} - \frac{1}{r^3}\right) - 2\pi G r_0^2 \rho_0^2 \left(\frac{1}{R^2} - \frac{1}{r^2}\right) \end{aligned}$$

For $r < r_0$,

$$\begin{aligned}
P - P_{r>r_0} &= \int \mathrm{d}P \\
&= \int_r^{r_0} -\frac{GM(r)}{r^2}\rho(r)\mathrm{d}r \\
&= \int_r^{r_0} -\frac{4}{3}\pi r^3 \rho_0^2 \frac{G}{r^2}\mathrm{d}r \\
&= -\frac{4}{3}\pi\rho_0^2 G \int_r^{r_0} r\mathrm{d}r \\
&= -\frac{2}{3}\pi G\rho_0^2(r^2 - r_0^2)
\end{aligned}$$

To conclude,

$$\therefore P(r) = \begin{cases} -\frac{2}{3}\pi G\rho_0^2(r^2 - r_0^2) + \frac{8}{9}\pi G r_0^3 \rho_0^2 \left(\frac{1}{R^3} - \frac{1}{r^3}\right) - 2\pi G r_0^2 \rho_0^2 \left(\frac{1}{R^2} - \frac{1}{r^2}\right) & \text{for } r < r_0, \\ \frac{8}{9}\pi G r_0^3 \rho_0^2 \left(\frac{1}{R^3} - \frac{1}{r^3}\right) - 2\pi G r_0^2 \rho_0^2 \left(\frac{1}{R^2} - \frac{1}{r^2}\right) & \text{for } r_0 \leq r \leq R, \\ 0 & \text{for } r > R. \end{cases}$$

Problem 16 - 12

The following table gives one model for the future evolution of the Sun:
For each future time, find the effective temperature of the Sun. Then plot an evolutionary track on an H-R diagram.

Table 2. Table for Problem 12

Time (Gy)	**Luminosity ($L_\odot$)**	**Radius ($R_\odot$)**
5.5	1.08	1.04
6.6	1.19	1.08
7.7	1.32	1.14
8.8	1.50	1.22
9.8	1.76	1.36

Solution

Since both luminosity and the radius is given, the temperature can be determined:

$$L = 4\pi R^2 \sigma T^4$$

Let's rewrite the table.

Table 3. Table After Calculating Temperature

Index	Time (Gy)	Luminosity ($L_\odot$)	Radius ($R_\odot$)	Temperature (T)
1	5.5	1.08	1.04	5862 K
2	6.6	1.19	1.08	5866 K
3	7.7	1.32	1.14	5875 K
4	8.8	1.50	1.22	5887 K
5	9.8	1.76	1.36	5902 K

To plot this table into a graph, with log scale applied for both of the axis and x-axis reversed just like H-R diagram, we get the following figure.

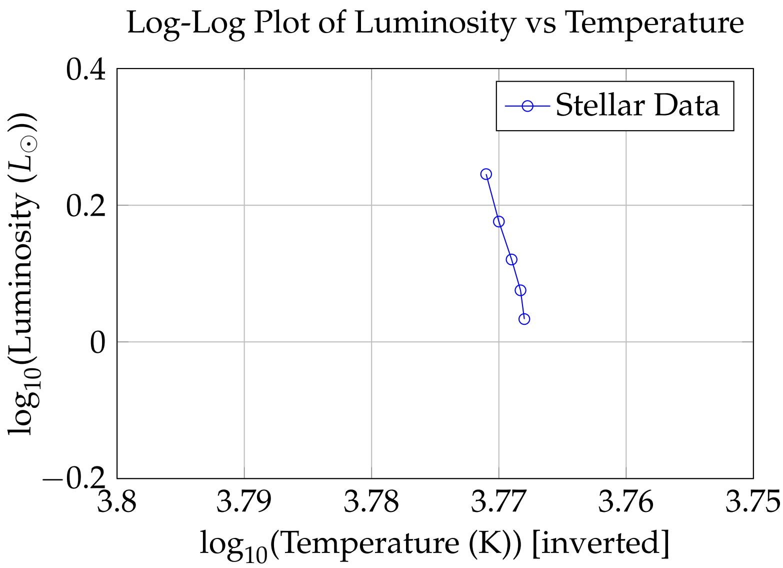


Figure 1: Log-log plot of Luminosity vs Temperature

The graph shows almost vertically increasing luminosity while the effective temperature nearly stays to same. This can either imply evolutionary stage or the helium burning onset stage for the star, where the luminosity highly increases compared to the temperature.

Problem Set

Stars and Universe
September 25, 2024
21-123 황동우, 23-017 김민겸, 23-050 박서진, 23-119 최원

GROUP 4

Problem 1

1.1

Let's define radial vectors of masses m_1 and m_2 to be r_1 and r_2, respectively. Then, relative position of m_2 to m_1 is

$$\boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1$$

From the Newton's law of gravitation, gravitational force f_1 that m_1 experiences from m_2 is

$$f_1 = \frac{Gm_1m_2}{r^3}\boldsymbol{r} = m_1\ddot{r}_1$$

and vice versa for f_2, which is

$$f_2 = -\frac{Gm_1m_2}{r^3}\boldsymbol{r} = m_2\ddot{r}_2$$

By defining a new symbol $\mu \equiv G(m_1 + m_2)$, the equation for the relative position is

$$\ddot{r}_2 - \ddot{r}_1 = \ddot{r} = -\frac{\boldsymbol{r}}{r^3}G(m_1 + m_2) = -\frac{\mu}{r^3}\boldsymbol{r}$$

Therefore, the relative force $\boldsymbol{f}$ on the celestial body m_2 is

$$\boldsymbol{f} = -\frac{\mu m_2}{r^3}\boldsymbol{r}$$

1.2

It is possible to derive the equation of relative motion of the celestial body by integrating the formula for $\ddot{r}$. However, a simpler method will be used in this proof, starting from the conservation of angular momentum. The orbital angular momentum vector $\boldsymbol{h}$ is

$$\boldsymbol{h} = \boldsymbol{r} \times \dot{\boldsymbol{r}}$$

$$\dot{\boldsymbol{h}} = \boldsymbol{r} \times \ddot{\boldsymbol{r}} + \dot{\boldsymbol{r}} \times \dot{\boldsymbol{r}} = \boldsymbol{r}\left(-\frac{\mu \boldsymbol{r}}{r^3}\right) = 0$$

The physical meaning of this result is that the orbital angular momentum vector with in a gravitational field remains constant. Next, the orbital angular momentum vector can be written as

$$\boldsymbol{h} \times \ddot{\boldsymbol{r}} = (\boldsymbol{r} \times \dot{\boldsymbol{r}}) \times \left(-\frac{\mu \boldsymbol{r}}{r^3}\right) = -\frac{\boldsymbol{r} \cdot \boldsymbol{r}\mu}{r^3}\dot{\boldsymbol{r}} + \frac{\dot{r}\mu \boldsymbol{r}}{r^3}\boldsymbol{r} = -\left(\frac{\mu \dot{\boldsymbol{r}}}{r} - \frac{\mu \dot{r}}{r^2}\boldsymbol{r}\right) = -\mu \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{r}}{r}\right)$$

Since the angular momentum $\boldsymbol{h}$ remains constant regardless of time,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{h} \times \dot{\boldsymbol{r}} + \mu \frac{\boldsymbol{r}}{r}\right) = 0$$

Integrating this gives us

$$\boldsymbol{h} \times \dot{\boldsymbol{r}} + \mu \frac{\boldsymbol{r}}{r} = -\mu \boldsymbol{e} = \text{constant}$$

Where $\boldsymbol{e}$ is a constant of integration. By multiplying $\boldsymbol{r}$ on both sides, we get:

$$\mu(r + \boldsymbol{e} \cdot \boldsymbol{r}) = (\dot{r} \times \boldsymbol{h}) \cdot \boldsymbol{r} = (\boldsymbol{r} \times \dot{r}) \cdot \boldsymbol{h} = h^2$$

Therefore, the equation of relative motion of the celestial body is

$$r + \boldsymbol{e} \cdot \boldsymbol{r} = \frac{h^2}{\mu}$$

1.3

Since $\boldsymbol{e} \cdot \boldsymbol{r} = er\cos\theta$, the result from 1.2 can be rewritten as

$$r + er\cos\theta = \frac{h^2}{\mu}$$

$$r = \frac{h^2/\mu}{1 + e\cos\theta}$$

Which is the final form we needed.

Problem 2

Although problem 1.3 already gave us the orbit equation, there is still a need for an additional geometrical interpretation. The given equation

$$r = \frac{h^2/\mu}{1 + e\cos\theta}$$

is not only the orbit equation we derived, but also the general formula of the conic sections where e stands for eccentricity of the given curve.

When $\theta = 0$, r becomes minimum. This means that the direction of the eccentricity vector $\boldsymbol{e}$ points toward perihelion. We can also observe that $r = h^2/\mu$ when $\theta = \pi/2$. This length of r when $\theta = \pi/2$ is called 'semi latus rectum, (p)'.
Therefore, the formula for orbit is now simplified as

$$r = \frac{p}{1 + e\cos\theta}$$

And the following stands:

i) $e = 0$: circle

$$r = p$$

ii) $0 < e < 1$: ellipse

$$r' + p = 2a$$
$$p = a(1 - e^2)$$
$$r = \frac{a(1-e^2)}{1 + e\cos\theta}$$

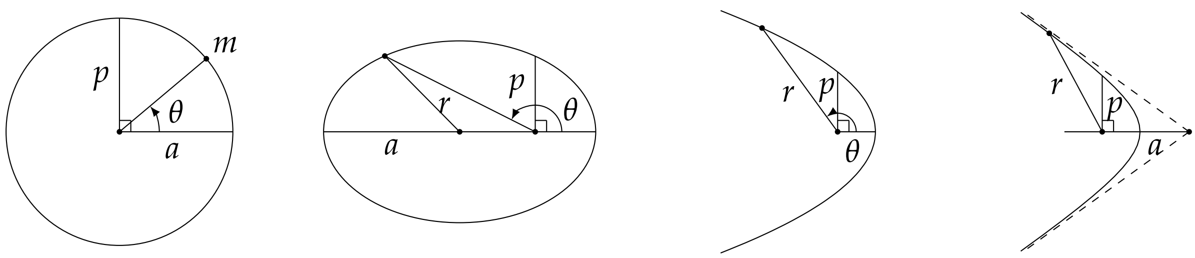
iii) $e = 1$: parabola

$$p = 2q$$
$$r = \frac{2q}{1 + \cos\theta}$$

iv) $e > 1$: hyperbola

$$r' - p = 2a$$
$$p = a(e^2 - 1)$$
$$r = \frac{a(e^2-1)}{1 + \cos\theta}$$

Where $a = \dfrac{h^2/\mu}{1 - e^2}$.



Circle Ellipse Parabola Hyperbola

Figure 1: Conic Section

Problem 3

3.1

From the gravitational force equation we derived earlier, the work done is

$$\int_A^B \boldsymbol{f} \cdot \mathrm{d}\boldsymbol{r} = \int_A^B \left(-m_2 \frac{\mu}{r^2}\hat{\boldsymbol{r}}\right) \cdot \mathrm{d}\boldsymbol{r} = \frac{\mu m_2}{r_B} - \frac{\mu m_2}{r_A}$$

It can be seen that the work done with in a gravitational field is only dependent to the distance between points A and B.

3.2

If we apply the Newton's law of motion $F = ma$, we get

$$W = \int_A^B \boldsymbol{f} \cdot \mathrm{d}\boldsymbol{r} = \int_A^B m_2 \frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} \cdot \mathrm{d}\boldsymbol{r}$$

$$= \int_A^B m_2 \frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} \cdot \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \mathrm{d}t = \frac{m_2}{2} \int_A^B \frac{\mathrm{d}}{\mathrm{d}t} (\boldsymbol{V} \cdot \boldsymbol{V}) \mathrm{d}t = \frac{m_2}{2} \int_A^B \mathrm{d}\left(\boldsymbol{V}^2\right)$$

Therefore,

$$W = \frac{1}{2} m_2 v_B^2 - \frac{1}{2} m_2 v_A^2$$

By using the result of 3.1, we can conclude that the total energy at points A and B follows the relationship

$$T = \frac{1}{2} m_2 v_A^2 - \frac{\mu m_2}{r_A} = \frac{1}{2} m_2 v_B^2 - \frac{\mu m_2}{r_B}$$

That is, the total energy T is constant in the gravitational field.

Problem 4

4.1

The relative velocity of celestial object m_2 in the r direction, denoted by $\boldsymbol{V}_r$, can be expressed as the sum of its normal and tangential velocities $\boldsymbol{V}_r$ and $\boldsymbol{V}_\theta$

$$\boldsymbol{V} = \boldsymbol{V_r} + \boldsymbol{V_\theta}$$

$$\boldsymbol{V_\theta} = r\dot{\theta}\hat{\boldsymbol{\theta}}, \quad \boldsymbol{V_r} = \dot{r}\hat{\boldsymbol{r}}$$

The areal velocity $\dot{A}$ is given by

$$A = \frac{1}{2} r V_t = \frac{1}{2} r^2 \dot{\theta} = \frac{1}{2} h$$

Using this, the relative velocity is

$$V^2 = \frac{h^2}{p} \left(\frac{2}{r} - \frac{1 - e^2}{p} \right)$$

which can also be written as

$$V^2 = \mu \left(\frac{2}{r} - \frac{1 - e^2}{p} \right)$$

Thus, the Vis-viva equation is derived.

4.2

By multiplying the above equation by the mass m_2 and using energy equation $T = \frac{1}{2}m_2 v^2 - \frac{\mu m_2}{r}$, we obtain

$$\frac{1}{2}m_2 V^2 - \frac{\mu m_2}{r} = -\frac{\mu m_2}{2p}\left(1 - e^2\right) = \frac{\mu^2 m_2}{2h^2}\left(1 - e^2\right) = T$$

where T represents the total energy of the system. This shows that the vis-viva equation is deeply related with the total energy of mass m_2. It also coincides with the orbital eccentricity equation which will be discussed in the following section

$$\mu^2\left(e^2 - 1\right) = 2h^2 k, \quad e = \left[1 + \frac{2h^2 T}{\mu^2 m_2^2}\right]^{1/2}$$

Problem 5

5.1

Let the distance at perigee be denoted by r_p. Since $\theta = 0$ at perigee, we can obtain

$$r_p = \frac{h^2}{\mu(1+e)}$$

At this point, the total energy T of the object m_2 is the sum of its kinetic and potential energy

$$T = \frac{1}{2}m_2 V_p^2 - \frac{\mu m_2}{r_p}$$

The velocity at perigee V_p is given by

$$V_p = \frac{h}{r_p}$$

Combining these equations above to eliminate r_p, we obtain the following expression for the eccentricity

$$e = \left(1 + \frac{2h^2 T}{\mu^2 m_2^2}\right)^{1/2}$$

Thus, the eccentricity e is determined by angular momentum, total energy, and mass.

5.2

The characteristics are as follows.

Orbital Type	e	p	$\dot{A}$	h	T	V^2
Circle	0	a	$\frac{1}{2}\sqrt{\mu a}$	$\sqrt{\mu a}$	$-\frac{\mu m_2}{2a}$	$\frac{\mu}{a}$
Ellipse	$0 < e < 1$	$a(1-e^2)$	$\frac{1}{2}\sqrt{\mu a(1-e^2)}$	$\sqrt{\mu a(1-e^2)}$	$-\frac{\mu m_2}{2a}$	$\mu\left(\frac{2}{r}-\frac{1}{a}\right)$
Parabola	1	$2q$	$\sqrt{\frac{1}{2}\mu q}$	$\sqrt{2\mu q}$	0	$\frac{2\mu}{r}$
Hyperbola	$e > 1$	$a(e^2-1)$	$\frac{1}{2}\sqrt{\mu a(e^2-1)}$	$\sqrt{\mu a(e^2-1)}$	$\frac{\mu m_2}{2a}$	$\mu\left(\frac{2}{r}+\frac{1}{a}\right)$

Table 1. Orbital Parameters for Different Orbital Types

Problem 6

6.1

The angular momentum is $h = r \times \dot{r} = r^2\dot{\theta}$. The magnitude of h is $r^2\dot{\theta}$. The areal velocity is the area swept by the radial vector in time dt, which is represented as follows. Assume that the areal velocity is constant over time. The distance and the angle θ form the following equation

$$\dot{A} = \frac{1}{2}r^2\dot{\theta} = \frac{h}{2}$$

Which we have already discussed in the section 4.1. From the equation above, it can be shown that the angular momentum vector h remains constant in time, indicating that it is conserved. Therefore, the orbital motion vector sweeps out equal areas in equal times. Which indicates the Kepler's second law, that areal velocity of a planet remains constant.

6.2

As the Sun-planet distance changes, the orbital speed changes as well. According to Kepler's second law, a planet moves faster when it is closer to the Sun (perihelion) and slower when it is farther from the Sun (aphelion).
From the areal velocity formula, the following equation can be derived:

$$dA = \frac{1}{2}h\,dt$$

Integrating over the complete orbital period gives the following

$$\int_0^P dA = \frac{1}{2}h\int_0^P dt$$

Here, P is the orbital period. The area of eclipse is as follows

$$\pi ab = \pi a^2\sqrt{1-e^2}, \quad b = a\sqrt{1-e^2}, \quad r_{ab} = \frac{a\left(1-e^2\right)}{1+e\cos\theta}$$

a and b are the semi-minor and semi-major axis of the elliptical orbit, and e is eccentricity. The relationship is given by the following

$$\pi a^2\sqrt{1-e^2} = \frac{1}{2}hP$$

$$h = \sqrt{\mu p} = \sqrt{G(m_1+m_2)a\left(1-e^2\right)}$$

Combining these two equations yields:

$$P^2 = \frac{4\pi^2}{G(m_1+m_2)}a^3$$

This equation is derived from Newton's laws and matches Kepler's third law. This demonstrates that Kepler's laws are consistent with Newtonian mechanics.

Problem 7

Firstly, the Semi-Major axis (a) is the average distance between the two bodies. For elliptical orbits, it is the longest axis that spans the ellipse. For circular orbits, it is simply the radius.

Secondly, Eccentricity (e) measures the shape of the orbit. It determines how much the orbit deviates from being circular.

Next, Inclination (i) is the angle between the orbital plane of the body and a reference plane (usually the equatorial plane of the primary body or the ecliptic plane for orbits around the Sun). It defines how tilted the orbit is relative to this reference plane.

Also, Longitude of the Ascending Node (Ω) is the angle from a reference direction (usually the vernal equinox) to the ascending node, which is the point where the orbiting body crosses the reference plane going north.

And then, Argument of Periapsis (ω) defines the orientation of the ellipse within the orbital plane. It is the angle between the ascending node and the point of closest approach (periapsis).

Finally, True Anomaly τ is the angle between the periapsis and the current position of the orbiting body. It changes over time as the body moves along its orbit.

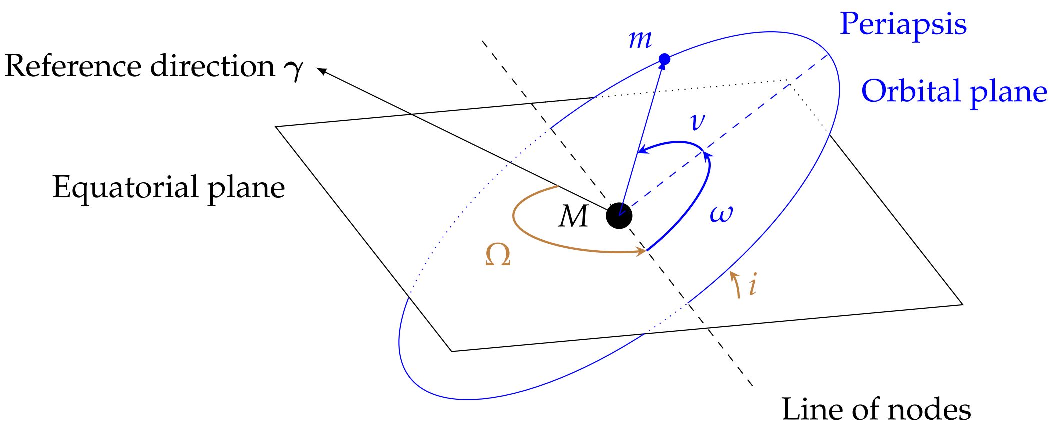


Figure 2: Orbital elements

These six elements $(a, e, i, \Omega, \omega, \tau)$ describe the shape, size, and orientation of the orbit, as well as the position of the celestial body along the orbit at a specific time.

Problem 8

Let the unit vectors in the x, y, and z-axis directions in the ecliptic plane be $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$, and the unit vectors in the x', y', and z'-axis directions in the orbital plane be $\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}$, the relationship between these two coordinate systems is

$$\begin{bmatrix} \boldsymbol{e_1} \\ \boldsymbol{e_2} \\ \boldsymbol{e_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos i & \sin i \\ 0 & -\sin i & \cos i \end{bmatrix} \begin{bmatrix} \cos\Omega & \sin\Omega & 0 \\ -\sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{i} \\ \boldsymbol{j} \\ \boldsymbol{k} \end{bmatrix}$$
$$= \begin{bmatrix} \cos\Omega\boldsymbol{i} + \sin\Omega\boldsymbol{j} \\ -\cos i \sin\Omega\boldsymbol{i} + \cos i \cos\Omega\boldsymbol{j} + \sin i\boldsymbol{k} \\ \sin i \sin\Omega\boldsymbol{i} - \sin i \cos\Omega\boldsymbol{j} + \cos i\boldsymbol{k} \end{bmatrix}$$

At this time, i is the orbital inclination, and Ω is the ascending node longitude.
Since the position vectors given in the problem are all on the orbital plane, the cross product of any two vectors is parallel to $\boldsymbol{e_3}$, which is perpendicular to the orbital plane.

$$\boldsymbol{e_3} = \frac{\boldsymbol{r_1} \times \boldsymbol{r_3}}{\|\boldsymbol{r_1} \times \boldsymbol{r_3}\|} = \frac{1}{C}\left[(y_1 z_3 - z_1 y_3)\boldsymbol{i} + (z_1 x_3 - x_1 z_3)\boldsymbol{j} + (x_1 y_3 - x_3 y_1)\boldsymbol{k}\right]$$
$$C = \|\boldsymbol{r_1} \times \boldsymbol{r_3}\| = \left[(y_1 z_3 - z_1 y_3)^2 + (z_1 x_3 - x_1 z_3)^2 + (x_1 y_3 - x_3 y_1)^2\right]^{1/2}$$

If we compare and summarize the coefficients of the unit vectors,

$$i = \arccos \frac{(x_1 y_3 - x_3 y_1)}{C}$$
$$\Omega = \arctan \left(\frac{y_1 z_3 - z_1 y_3}{z_1 x_3 - x_1 z_3} \right)$$

From the elliptical orbit equation $r = \frac{a(1-e^2)}{1+e\cos\theta}$ in polar coordinates, we get:

$$e[(\boldsymbol{r} \cdot \boldsymbol{e_1}) \cos\omega + (\boldsymbol{r} \cdot \boldsymbol{e_2}) \sin\omega] = a(1-e^2) - r$$

(where $u \equiv \omega + \theta, r\cos u = \boldsymbol{r} \cdot \boldsymbol{e_1}, r \sin u = \boldsymbol{r} \cdot \boldsymbol{e_2}$)
Applying Equation of Orbit to $\boldsymbol{r_1}, \boldsymbol{r_2}, \boldsymbol{r_3}$, we get:

$$e\cos\omega[(\boldsymbol{r_1} - \boldsymbol{r_2}) \cdot \boldsymbol{e_1}] + e\sin\omega[(\boldsymbol{r_1} - \boldsymbol{r_2}) \cdot \boldsymbol{e_2}] = r_2 - r_1$$
$$e\cos\omega[(\boldsymbol{r_1} - \boldsymbol{r_3}) \cdot \boldsymbol{e_1}] + e\sin\omega[(\boldsymbol{r_1} - \boldsymbol{r_3}) \cdot \boldsymbol{e_2}] = r_3 - r_1$$

If we write $e\cos\omega, e\sin\omega$ through $A_1, A_2, andD$,

$$e\cos\omega = \frac{A_1}{D}, e\sin\omega = \frac{A_2}{D}$$

$$A_1 = [(r_2 - r_3)\boldsymbol{r_1} + (r_3 - r_1)\boldsymbol{r_2} + (r_1 - r_2)\boldsymbol{r_3}] \cdot \boldsymbol{e_2}$$
$$A_2 = -[(r_2 - r_3)\boldsymbol{r_1} + (r_3 - r_1)\boldsymbol{r_2} + (r_1 - r_2)\boldsymbol{r_3}] \cdot \boldsymbol{e_1}$$
$$D = \det \left(\begin{bmatrix} (\boldsymbol{r_1} - \boldsymbol{r_2}) \cdot \boldsymbol{e_1} & (\boldsymbol{r_1} - \boldsymbol{r_2}) \cdot \boldsymbol{e_2} \\ (\boldsymbol{r_1} - \boldsymbol{r_3}) \cdot \boldsymbol{e_1} & (\boldsymbol{r_1} - \boldsymbol{r_3}) \cdot \boldsymbol{e_2} \end{bmatrix} \right)$$

From the solution of system of equations and orbital equation of ellipse, the orbital eccentricity and perihelion anomaly can be obtained as follows.

$$e = \frac{\sqrt{A_1^2 + A_2^2}}{D}$$
$$w = \arctan \frac{A_2}{A_1}$$
$$a = \frac{rD + (\boldsymbol{r} \cdot \boldsymbol{e_1})A_1 + (\boldsymbol{r} \cdot \boldsymbol{e_2})A_2}{D(1-e^2)}$$

The Eccentric Anomaly E is $r = 1 + e\cos E$, $E = \arccos \dfrac{r-1}{e}$, and let the Mean Anomaly M. Since $M = E - e\sin E$, τ is

$$\tau = t - \frac{(E - e\sin E)P}{2\pi}$$

(where $M = \dfrac{2\pi}{P}(t - \tau)$)
As shown above, all orbital elements can be expressed as three position vectors.

Problem 9

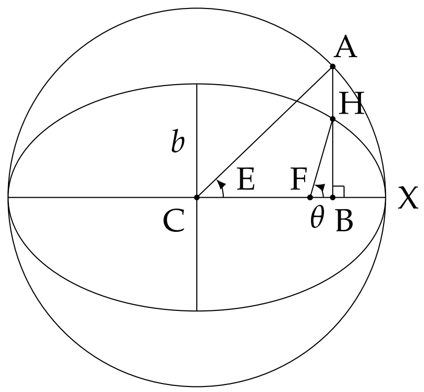


Figure 3: Eccentric Anomaly and True Anomaly

Using the relationship between Eccentric Anomaly(E) and True Anomaly(θ) in Figure 2,

$$\begin{aligned} r\cos\theta &= a\cos E - ae \\ r\sin\theta &= \overline{HB} = \overline{AB}\cdot\sqrt{1-e^2} = a\sqrt{1-e^2}\sin E \\ \boldsymbol{r} &= a(\cos E - e)\boldsymbol{i} + a\sqrt{1-e^2}\sin E\boldsymbol{j} \\ &= a(\cos E - e)\boldsymbol{i} + b\sin E\boldsymbol{j} \end{aligned}$$

In Fig 2, the area (A) of FHX can be obtained with Kepler's second law and geometry, respectively. At this time, the FHX of the ellipse is obtained by multiplying the corresponding area of the circumscribed circle by the ratio of the axes, so it can be obtained as the difference between the area of CAX and the triangle CAF.

$$\begin{aligned} A &= \frac{b}{a}\left(\frac{1}{2}a^2E - \frac{1}{2}ae\cdot a\sin E\right) = \frac{1}{2}ab(E - e\sin E) \\ &= \pi ab\frac{t-\tau}{P} \end{aligned}$$

By organizing this, the desired $E - e\sin E = M$ is derived.

Problem 10

The Potential Energy (U), of a celestial body m_2 in circular motion of radius a is $-\dfrac{\mu m_2}{a}$. Additionally, since gravity acts as a centripetal force, Kinetic Energy (K), is $\dfrac{1}{2}\dfrac{\mu m_2}{a}$. Therefore, since there is a constant multiple relationship between Potential Energy and Kinetic Energy, the Virial Theorem is established.

Problem 11

The average value of Kinetic Energy over time of a celestial body m_2 can be obtained using Kepler's Equation.

$$E - e\sin E = \frac{2\pi}{P}(t-\tau)$$

Differentiating both sides of the above equation and using the chain rule,

$$(1-e\cos E)\mathrm{d}E = \frac{2\pi}{P}\mathrm{d}t$$

$$< K >= \frac{\int_0^P \frac{1}{2}m_2 v^2 \mathrm{d}t}{\int_0^P \mathrm{d}t} = \frac{\frac{1}{2}m_2 \int_0^{2\pi} \frac{\mu}{a(1-e\cos E)}(1-e\cos E)\frac{P}{2\pi}\mathrm{d}E}{P} = \frac{1}{2}\frac{\mu m_2}{a}$$

Similarly, if we calculate Potential Energy,

$$< P >= \frac{\int_0^P -\frac{\mu m_2}{r}\mathrm{d}t}{\int_0^P \mathrm{d}t} = \frac{-\mu m_2 \int_0^{2\pi} \frac{1}{a(1-e\cos E)}(1-e\cos E)\frac{P}{2\pi}\mathrm{d}E}{P} = -\frac{\mu m_2}{a}$$

Therefore, it can be seen that the virial theorem holds even in elliptical orbital motion.

Problem 12

(a) By the law of conservation of mechanical energy, the escape velocity (v_e) is $\sqrt{\frac{2\mu}{R}} = \sqrt{\frac{2G(m_1+m_2)}{R}}$. If the probe's mass is very small, it does not matter if it is expressed as $v_e = \sqrt{\frac{2GM}{R}}$. Substituting the Earth's mass $M = 5.9722 \times 10^{29} kg$ into the equation, $v_e \approx 11.2 km/s$
(b) Substituting the orbital period $P = \frac{2\pi R}{v_c}$ as a function of the orbital speed (v_c) into Kepler's third law,

$$P^2 = \frac{4\pi^2 R^2}{v_c^2} = \frac{4\pi^2 R^3}{G(m_1+m_2)}$$

Therefore, $v_c = \sqrt{\frac{G(m_1+m_2)}{R}}$, $v_e = \sqrt{2}v_c$ holds. The Earth's orbital speed is about $29.8 km/s$, so $v_e' \approx 42 km/s$.

Problem 13

The satellite makes a quadratic curve orbit as the following formula.

$$r = \frac{h^2/\mu}{1+e\cos\theta}$$

μ, the standard gravitational parameter is $G(M+m)$. However since $m \ll M$, we can express μ as GM. The angular momentum h conserved throughout the entire orbit can be calculated by the initial condition, $h = v_0(R+H)$. Therefore we substitute these formulas to equation .

$$r = \frac{v_0^2 (R+H)^2}{(1+e\cos\theta)GM}$$

The perigee is at when $\theta = 0$, where the radial distance is expressed by the following.

$$r_p = \frac{v_0^2 (R+H)^2}{(1+e)GM} = a(1-e)$$

We manipulate formulas to obtain the eccentricity.
The specific orbital energy is expressed by $\epsilon = \frac{v^2}{2} - \frac{\mu}{r}$. Also, we know the Vis-Viva equation which is $v^2 = \mu(\frac{2}{r} - \frac{1}{a})$. Therefore we substitute the velocity terms of angular momentum, and the ϵ using Vis-Viva equation.

$$\epsilon = \frac{1}{2}\left(\mu\left(\frac{2}{r} - \frac{1}{a}\right)\right) - \frac{\mu}{r} = -\frac{\mu}{2a}$$

Therefore we can express the semi major axis (a) with ϵ and μ.
Since angular momentum $h = r^2\dot{\theta}$, we can substitute r from the ellipse equation.

$$h = \left(\frac{a\left(1-e^2\right)}{1+ecos\theta}\right)^2 \dot{\theta}$$

The velocity in the specific orbital energy can be substituted to $v^2 = \dot{r}^2 + r^2\dot{\theta}^2$, which gives:

$$v^2 = \left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + \left(r^2\dot{\theta}\right)^2$$

substitute $h = r^2\dot{\theta}$ and rearrange to express ϵ in terms of h.

$$\epsilon = \frac{1}{2}\left(\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + \left(\frac{h}{r}\right)^2\right) - \frac{\mu}{r}$$

From this manipulation we can show the following equation about orbital eccentricity.

$$e = \sqrt{1 + \frac{2\epsilon h^2}{\mu^2}}$$

13.1

Initial velocity $\boldsymbol{V_0} = 4\,\mathrm{km/s}$
$e = 0.740$
$r_p = 966\mathrm{km}$
In this condition, the r_p is smaller than $R+H$, which implies that the satellite's orbit decays significantly below its initial launch altitude, potentially leading to a re-entry or collision with the Earth's atmosphere

13.2

Initial velocity $\boldsymbol{V_0} = 7.85\,\mathrm{km/s}$
$e = 4.95 \cdot 10^{-4}$
$r_p = 6471\mathrm{km}$
$7.85km/s$ is very close to the orbital velocity required for a low Earth orbit (LEO), which is approximately 7.8 km/s at an altitude of about 200 km above the Earth's surface. At this speed, the satellite is likely to enter a stable, nearly circular orbit

13.3

Initial velocity $\boldsymbol{V_0} = 11\,\mathrm{km/s}$
$e = 0.965$
$r_p = 6471\mathrm{km}$
This velocity is insufficient to escape earth gravity, which should reach approximately $11.2km/s$. This will result in an highly eccentric elliptical orbit with a significantly higher apoapsis (farthest point from Earth).

13.4

Initial velocity $\boldsymbol{V_0} = 12\,\mathrm{km/s}$
$e = 1.34$
$r_p = 6471\mathrm{km}$
$12km/s$ is above Earth's escape velocity, meaning that the satellite will not be bound to an orbit around Earth but will instead follow a hyperbolic trajectory, escaping Earth's gravitational influence.

Problem 14

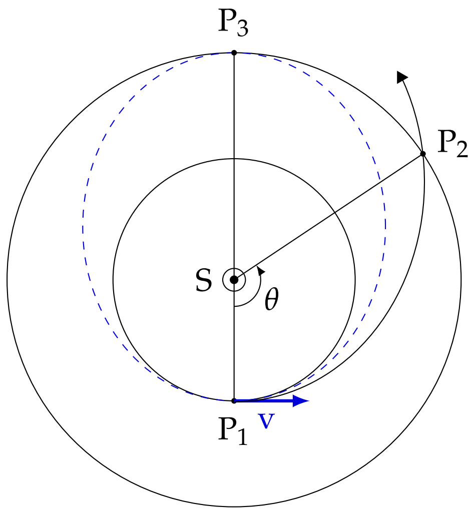


Figure 4: Interplanetary Satellite Orbit

First of all, we have to calculate the r_1 (initial radial distance of satellite) and r_2 (radial distance when satellite meets Mars).

$$r_1 = \left.\frac{a\left(1-e^2\right)}{1+e\cos\theta}\right|_{\theta=0} = a\left(1-e\right)$$

$$r_2 = \frac{a\left(1-e^2\right)}{1+e\cos\theta} = r_1\left(1+e\right)$$

This satisfies the relationship $r_2\left(1+e\cos\theta\right) = r_1\left(1+e\right)$ between r_1 and r_2. Then, we express the eccentricity as the following.

$$e = -\frac{r_1 - r_2}{r_1 - r_2\cos\theta}$$

Plugging this into $a = \dfrac{r_1}{1-e}$, we get the following expression for a.

$$a = \frac{r_1\left(r_1 - r_2\cos\theta\right)}{2r_1 - r_2\left(1+\cos\theta\right)}$$

Next, we want to calculate the Δvelocity for the launch velocity and the Earth revolution velocity.

$$\Delta\boldsymbol{v} = \boldsymbol{v_f} - \boldsymbol{v_E} = \sqrt{\mu}\left(\left(\frac{2}{r_1}-\frac{1}{a}\right)^{\frac{1}{2}} - \left(\frac{2}{r}-\frac{1}{a}\right)^{\frac{1}{2}}\right)$$

We go through the following calculation steps to organize this equation. ($a_E = 1\,\text{AU}$)

$$\sqrt{\mu}\left(\left(\frac{2}{r_1}-\frac{1}{a}\right)^{\frac{1}{2}} - \left(\frac{2}{r}-\frac{1}{a_E}\right)^{\frac{1}{2}}\right)$$

$$= \sqrt{\mu}\left(\left(\frac{2}{r_1}-\frac{2r_1 - r_2\left(1+\cos\theta\right)}{r_1\left(r_1 - r_2\cos\theta\right)}\right)^{\frac{1}{2}} - \left(\frac{2}{r_2}-1\right)^{\frac{1}{2}}\right)$$

$$= \sqrt{\mu} \left(\sqrt{\frac{r_2}{r_1}} \left(\frac{1 - \cos\theta}{r_1 - r_2 \cos\theta} \right)^{\frac{1}{2}} - \left(\frac{2}{r_2} - 1 \right) \right)$$

Therefore, we can organize Δv as the following.

$$\Delta v = \sqrt{\mu} \left(\sqrt{\frac{r_2}{r_1}} \left(\frac{1 - \cos\theta}{r_1 - r_2 \cos\theta} \right)^{\frac{1}{2}} - \left(\frac{1}{r_1} - 1 \right)^{\frac{1}{2}} \right)$$

Since v_E is 30 km/s, we can express the launch velocity as $v_f = 30\,\mathrm{km/s} + \Delta v$.
We want to make Δv minimum for a Hohmann orbit.
If we look at the Δv expression, we can know that the θ-dependent term is only $\frac{1 - \cos\theta}{r_1 - r_2 \cos\theta}$.
Therefore, we differentiate this term with respect to θ and find when it becomes 0 to find its minimum.

$$\frac{\mathrm{d}}{\mathrm{d}\theta} \frac{1 - \cos\theta}{r_1 - r_2 \cos\theta} = \frac{\sin\theta\,(r_1 - r_2 \cos\theta) - (1 - \cos\theta)\,(r_2 \sin\theta)}{(r_1 - r_2 \cos\theta)^2}$$
$$= \frac{\sin\theta\,(r_1 - r_2)}{(r_1 - r_2 \cos\theta)^2}$$

To make this zero, the $\sin\theta$ term needs to be zero, which gives us two solutions: $\theta = 0$ and $\theta = \pi$ [rad]. However, to make the actual energy the minimum, the answer we want is only $\theta = \pi$ [rad]. This launch angle is for the Hohmann orbit.
In this orbit, $a = \frac{1}{2}(r_1 + r_2)$ and the period becomes $P = \left(\frac{1 + r_2}{2} \right)^{3/2}$ [yrs].

Problem 15

15.1

The r_1 is 1 AU, which is the radius of Earth's orbit, and the r_2 is 1.52 AU, which is the radius of Mars' orbit.

$$a = \frac{r_1 - r_2}{2} = 1.26\,\mathrm{AU}$$

$$c = \frac{1}{2}(r_1 - r_2) = 0.26\,\mathrm{AU}$$

$$\therefore e = \frac{c}{a} = 0.206$$

$$P = \left(\frac{1 + r_2}{2} \right)^{3/2} = 1.41\,[\mathrm{yrs}]$$

Since this is the full orbital period, the time to travel from Earth to Mars (which is half the orbital period) is expressed as the following:

$$\frac{P}{2} = 0.705\,\text{yrs}$$

15.2

Since the travel time is approximately 0.705 yrs, we need Mars to be at a point where it will reach the correct position in that time.
We first calculate the angular velocity of Mars.

$$\omega_{\text{Mars}} = \frac{360^\circ}{1.88\,\text{yrs}} = 191.49^\circ/\text{yrs}$$

In 0.705 yrs, Mars will travel:

$$\theta = \omega_{\text{Mars}} \cdot 0.705\,\text{yrs} = 134.97^\circ$$

Therefore, when MER is launched, Mars needs to be at a position $180^\circ - 134.97^\circ = 45.03^\circ$ ahead of Earth in its orbit.

15.3

For a Hohmann transfer orbit, the velocity at the point of launch from Earth's orbit can be calculated using the Vis-Viva equation:

$$v = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)} = \sqrt{1 \times \left(\frac{2}{1} - \frac{1}{1.26}\right)} = 1.1\,\text{AU/yrs}$$

Since Earth's orbital velocity is approximately 1.1 AU/yrs, the MER's launch velocity relative to Earth will be:

$$v_{\text{launch}} = v - v_E = 1.1 - 1 = 0.1\,\text{AU/yrs}$$

This is the additional velocity needed for the MER to escape Earth's orbit and enter the transfer orbit toward Mars.

Problem Set

Stars and Universe
October 21, 2024
21-123 황동우, 23-017 김민겸, 23-050 박서진, 23-119 최원

GROUP 4

Problem 1

1.1

Let $\mathrm{d}E$ be the energy passing through a small area element $\mathrm{d}A$ within a time interval $\mathrm{d}t$, between frequencies ν and $\nu + \mathrm{d}\nu$. It is evident that we can express this as follows, where F_ν is defined as the flux density.

$$\mathrm{d}E = F_\nu \, \mathrm{d}A \, \mathrm{d}t \, \mathrm{d}\nu \tag{1}$$

1.2

Let $\mathrm{d}E$ be the energy passing through $\mathrm{d}A$ over a time interval $\mathrm{d}t$, between frequencies ν and $\nu + \mathrm{d}\nu$, within a small solid angle $\mathrm{d}\Omega$ in the direction of $\hat{k}$. It is evident that we can express this as follows, where I_ν is defined as the specific intensity.

$$\mathrm{d}E = I_\nu(\hat{\boldsymbol{k}}) \cos\theta \, \mathrm{d}A \, \mathrm{d}t \, \mathrm{d}\nu \, \mathrm{d}\Omega \tag{2}$$

Problem 2

Consider two arbitrary small area elements $\mathrm{d}A_1$ and $\mathrm{d}A_2$, perpendicular to the rays. Also, consider the set of rays B passing through these two area elements. The energy carried by B will be constant. That is, the following holds from (2).

$$\mathrm{d}E_1 = I_{\nu,1}(\hat{\boldsymbol{k}}) \cos\theta \, \mathrm{d}A_1 \, \mathrm{d}\Omega_1 \, \mathrm{d}t \, \mathrm{d}\nu = \mathrm{d}E_2 = I_{\nu,2}(\hat{\boldsymbol{k}}) \cos\theta \, \mathrm{d}A_2 \, \mathrm{d}\Omega_2 \, \mathrm{d}t \, \mathrm{d}\nu \tag{3}$$

Meanwhile, since all the energy passing through $\mathrm{d}A_1$ also passes through $\mathrm{d}A_2$, if we define the distance between the two area elements as r, we can write the following:

$$\mathrm{d}A_1 = \mathrm{d}\Omega_1 / r^2 \tag{4}$$
$$\mathrm{d}A_2 = \mathrm{d}\Omega_2 / r^2 \tag{5}$$

∴ From (3), (4), (5) we can get $I_{\nu,1} = I_{\nu,2}$, and it means that specific intensity doesn't depend on the distance.

Problem 3

(1) and (2) show that the flux density from all directions is given by the integral over all directions:

$$F_\nu = \int I_\nu \cos\theta \, \mathrm{d}\Omega \tag{6}$$

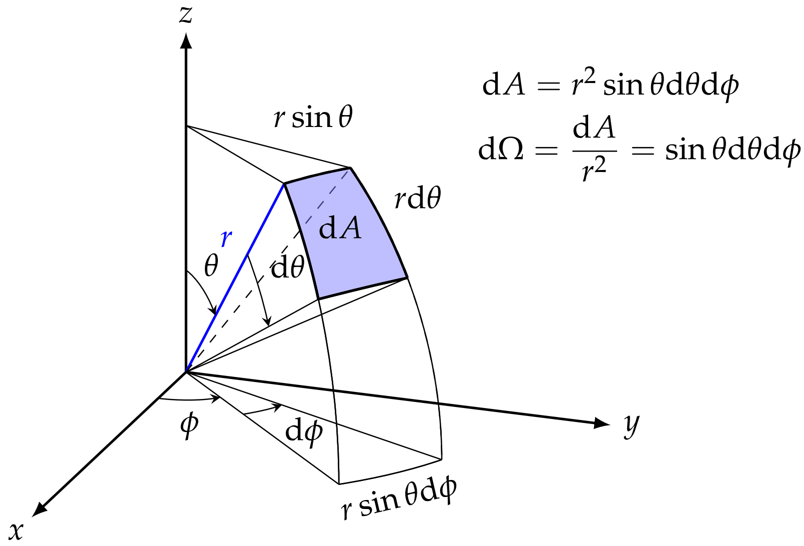


Figure 1: Solid Angle

For an isotropic radiation field generated by a source of radius R and intensity B_ν, the flux density can be expressed using (3) based on the fact shown in Figure (1), as follows:

$$F_\nu = \int_\Omega B_\nu \cos\theta \, \mathrm{d}\Omega \tag{7}$$
$$= \int_0^{\pi/2} \int_0^{2\pi} B_\nu \cos\theta \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \tag{8}$$
$$= \pi B_\nu \tag{9}$$

Problem 4

At a given point in space, the energy $\mathrm{d}E$ contained within a volume element of area $\mathrm{d}A$ and height $\mathrm{d}l$ is proportional to $\mathrm{d}A\,\mathrm{d}l\,\mathrm{d}\nu$, and the proportionality constant is defined as the radiation energy density u_ν.
The energy passing through area $\mathrm{d}A$ in time $\mathrm{d}t$ by a ray with specific intensity I_ν traveling in the direction of $\hat{k}$ is determined by (2). The time available for this ray to supply energy to a volume element of height $\mathrm{d}l$ is given by $\mathrm{d}t = \dfrac{\mathrm{d}l}{c\cos\theta}$. Thus, we can establish the following expression:

$$u_\nu = \frac{1}{c}\int I_\nu\,\mathrm{d}\Omega = \frac{I_\nu}{c}\int \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = \frac{4\pi}{c}I_\nu \tag{10}$$

Problem 5

The force $\mathrm{d}F$ exerted perpendicularly on an area element $\mathrm{d}A$ due to radiation will be proportional to $\mathrm{d}A\,\mathrm{d}\nu$. The proportionality constant is defined as the radiation pressure P_ν. The energy passing through $\mathrm{d}A$ during time $\mathrm{d}t$ by a ray with specific intensity I_ν in the direction of $\hat{k}$ is determined by (2). Since the energy of a photon is $h\nu$, the number of photons $\mathrm{d}N$ passing through $\mathrm{d}A$ is given by the following:

$$\mathrm{d}N = \frac{\mathrm{d}E}{h\nu} = \frac{I_\nu\cos\theta}{h\nu}\,\mathrm{d}A\,\mathrm{d}t\,\mathrm{d}\nu\,\mathrm{d}\Omega \tag{11}$$

Assuming that the surface absorbs the photons, the change in momentum $\mathrm{d}F'$ can be expressed using the photon momentum in the normal direction $\frac{h\nu\cos\theta}{c}$ as follows:

$$\mathrm{d}F' = \frac{h\nu\cos\theta}{c}\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{I_\nu\cos^2\theta}{c}\,\mathrm{d}A\,\mathrm{d}\nu\,\mathrm{d}\Omega \tag{12}$$

From (11), (12), and (10), the following can be calculated.

$$P_\nu = \int \frac{I_\nu\cos^2\theta}{c}\,\mathrm{d}\Omega \tag{13}$$
$$= \frac{I_\nu}{c}\int \cos^2\theta\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi \tag{14}$$
$$= \frac{4\pi}{3c}I_\nu = \frac{1}{3}u_\nu \tag{15}$$

Problem 6

Consider a cubic cavity heated to some temperature T. Assuming the cavity is a blackbody, light of all wavelengths inside will undergo continuous radiation and re-emission, following a specific energy distribution.

For a cube with side length L, nodes of the standing waves will form at the boundaries. That is, standing waves will be created.

$$L = \frac{\lambda}{2} j_i \tag{16}$$

Here, i is a constant representing the dimension's direction, and j_i refers to the number of half-wavelengths in 3-dimensional phase space. Therefore, the number of states of j between j and $j + \mathrm{d}j$, denoted as $g(j)$, is proportional to the volume of a spherical shell with radius j and thickness $\mathrm{d}j$, which is $4\pi j^2 \, \mathrm{d}j$.
By considering only the $1/8$ of the space where all components of j are positive, and accounting for two perpendicular polarization directions, we get:

$$g(j)\, \mathrm{d}j = 4\pi j^2 \cdot \frac{1}{8} \cdot 2 \, \mathrm{d}j = \pi j^2 \, \mathrm{d}j \tag{17}$$

Using (16), we know that $\mathrm{d}j = \frac{2L}{c} \, \mathrm{d}\nu$. Rewriting (17) in terms of ν yields:

$$g(\nu)\, \mathrm{d}\nu = \pi \left(\frac{2L\nu}{c} \right)^2 \frac{2L}{c} \, \mathrm{d}\nu = \frac{8\pi L^3}{c^3} \nu^2 \, \mathrm{d}\nu \tag{18}$$

Since the volume of the cavity is L^3, the density of waves with frequency ν per unit volume, $G(\nu)$, is:

$$G(\nu)\, \mathrm{d}\nu = \frac{1}{L^3} g(\nu) \, \mathrm{d}\nu = \frac{8\pi}{c^3} \nu^2 \, \mathrm{d}\nu \tag{19}$$

Now, consider there are N photons in total, where n_i photons have an energy of ϵ_i.

$$N = \sum_{i=1}^{k} n_i \tag{20}$$

$$E_{tot} = \sum_{i=1}^{k} \epsilon_i n_i \tag{21}$$

$$\overline{E} = \frac{E_{tot}}{N} \tag{22}$$

The task of finding the energy distribution of photons is equivalent to determining n_i as a function of E_i. The kinematic state of a photon can be represented by $\vec{r}$ and $\vec{p}$, but due to the uncertainty principle, the photon does not have a specific position but occupies a small volume in phase space. Photons are thus distributed across phase space in a variety of configurations, given by the 6 degrees of freedom (6DOF).
The total number of possible configurations W is:

$$W = \frac{N!}{n_1! n_2! \cdots n_k!} g_1^{n_1} g_2^{n_2} \cdots g_k^{n_k} \tag{23}$$

Even though the phase space has 6DOF, only one particle can occupy each cell. However, considering only $\vec{p}$, multiple photons with different $\vec{p}$ values can occupy the same cell.
Taking the natural logarithm of both sides of the equation for the total number of configurations W gives:

$$\ln W = \ln N! - \sum_{i=1}^{k} \ln n_i! + \sum_{i=1}^{k} n_i \ln g_i \tag{24}$$

Applying Stirling's approximation ($\ln n! = n \ln n - n$ for $n \gg 1$):

$$\ln W = N \ln N - N - \sum n \ln n + \sum n + \sum n \ln g \tag{25}$$
$$= N \ln N - \sum n \ln n + \sum n \ln g \tag{26}$$

Now, let us assume that the distribution which maximizes W is the most probable one. To apply the method of Lagrange multipliers, we further manipulate the equations:

$$\mathrm{d} \ln W_{max} = -\sum n \,\mathrm{d} \ln n - \sum \mathrm{d}n \ln g = 0 \tag{27}$$
$$-\sum n \,\mathrm{d} \ln n = -\sum n \frac{1}{n} \,\mathrm{d}n = -\sum \mathrm{d}n \tag{28}$$

Since both N and the total energy E_{tot} are fixed, we also have:

$$\sum \mathrm{d}n = 0 \tag{29}$$
$$\sum \epsilon \,\mathrm{d}n = 0 \tag{30}$$
$$-\sum \mathrm{d}n \ln n + \sum \mathrm{d}n \ln g = 0 \tag{31}$$

Adding $-\alpha\times$ (29) and $-\beta\times$ (30) to (31), and using the fact that $\mathrm{d}n = 0$, we obtain:

$$n(\epsilon) = g(\epsilon) A \exp\left\{-\frac{\beta p^2}{2m}\right\} \tag{32}$$

Thus, we can express the distribution in this form. Since the probability of a particle existing in a given volume of phase space is proportional to this distribution, $g(\epsilon)\,\mathrm{d}p = Ap^2\,\mathrm{d}p$. Combining this with (32), we get:

$$n(\epsilon) = Ap^2 \exp\left\{-\frac{\beta p^2}{2m}\right\} \tag{33}$$

Integrating both sides yields the total number of particles:

$$N = \int_0^\infty Cp^2 \exp\left\{-\frac{\beta p^2}{2m}\right\} \mathrm{d}p \tag{34}$$

According to the Gaussian integral, $C = N\sqrt{\frac{2}{\pi}}\left(\frac{\beta}{m}\right)^{\frac{3}{2}}$. Since the total energy $E_{tot} = \frac{3}{2}k_B T$, we have:

$$U = \int_0^\infty \frac{p^2}{2m} N \sqrt{\frac{2}{\pi}} \left(\frac{\beta}{m}\right)^{\frac{3}{2}} p^2 \exp\left\{-\frac{\beta p^2}{2m}\right\} \mathrm{d}p \tag{35}$$
$$= \frac{3N}{2\beta} \tag{36}$$

This distribution is known as the Maxwell-Boltzmann distribution.

Additionally, since energy is quantized, we have $\epsilon = nh\nu$. Now, let's calculate the average energy $\overline{\epsilon}$. By substituting $x = e^{-h\nu/k_BT}$, we proceed as follows:

$$\overline{\epsilon} = \frac{\sum \epsilon f_{MB}(\epsilon)}{\sum f_{MB}(\epsilon)} = \frac{\sum nh\nu f_{MB}(nh\nu)}{\sum f_{MB}(nh\nu)} \tag{37}$$
$$= \frac{0h\nu f_{MB}(0h\nu) + 1h\nu f_{MB}(1h\nu) + 2h\nu f_{MB}(2h\nu) + \cdots}{f_{MB}(0h\nu) + f_{MB}(1h\nu) + f_{MB}(2h\nu) + \cdots} \tag{38}$$
$$= \frac{h\nu(x + x^2 + \cdots)}{1 + x + x^2 + \cdots} \tag{39}$$
$$= \frac{h\nu\frac{x}{(1-x)^2}}{\frac{1}{x-1}} \tag{40}$$
$$\overline{\epsilon} = h\nu\frac{1}{\exp\{(h\nu/k_BT)\} - 1} \tag{41}$$

We call $\overline{\epsilon} = h\nu\frac{1}{\exp\{(h\nu/k_BT)\}-1}$ the Bose-Einstein distribution. The energy density $u(\nu, T)$ can then be expressed as follows:

$$u(\nu, T) = G(\nu)f_{BE}(\nu, T) \tag{42}$$
$$= \frac{8\pi\nu^2}{c^3}\frac{h\nu}{\exp\{(h\nu/k_BT)\} - 1} \tag{43}$$

Using (10), this can be further expressed as:

$$B(\nu, T) = \frac{2h\nu^3}{c^2}\frac{1}{\exp\{h\nu/k_BT\} - 1} \tag{44}$$

Problem 7

First, let's examine the Rayleigh-Jeans law. When $x \approx 0$, we can approximate $e^x \approx 1 + x$. Substituting this into (44), we have:

$$B(\nu, T) = \frac{2h\nu^3}{c^2}\frac{k_BT}{h\nu} = \frac{2k_BT}{\lambda^2} \tag{45}$$

Similarly, when x is very large, we can approximate $\frac{1}{e^x-1} \approx e^{-x}$. Substituting this into (44), we get:

$$B(\nu, T) = \frac{2h\nu^3}{c^2}e^{-\frac{h\nu}{k_BT}} \tag{46}$$

Problem 8

The effective temperature is defined as the temperature of a black body that emits the same amount of flux as the observed total flux, integrated over all wavelengths emitted from a star. The flux density received at a distance r can be expressed using the inverse square law:

$$F = \frac{L}{2\pi r^2} = \left(\frac{\alpha^2}{2}\right)\sigma T_{\text{eff}}^4 \tag{47}$$

Additionally, the temperature of a black body that emits radiation with the same ratio of intensity observed in two different wavelength ranges is referred to as the color temperature.

Problem 9

Bohr's first postulate states that "only orbits that satisfy quantum conditions are possible for electrons." The allowed orbits must have the angular momentum of the electron quantized as an integer multiple of $\hbar$. Assuming the electron orbits in a circular path, the centripetal force maintaining the electron's orbit is described by the Coulomb force between the electron and the nucleus.

$$m_e \frac{v^2}{r} = k\frac{Ze^2}{r^2} \tag{48}$$

$$r = \left(\frac{\hbar^2}{m_e k e^2 Z e}\right) h^2 \tag{49}$$

The total energy of these orbits is given by:

$$E = \frac{1}{2}m_2 v^2 - \frac{kZe^2}{r} = -\left(\frac{m_e k^2 e^4 Z e^2}{2\hbar^2}\right)\frac{1}{n^2} \tag{50}$$

Thus, it can be observed that the smallest Bohr orbit, with $n = 1$, is the most tightly bound, and all orbits remain bound until they approach $E \to 0$ as $n \to \infty$. When $E > 0$, unbound electron orbits become continuously possible.
Bohr's second postulate concerns quantized radiation. When an electron transitions from one orbit to another, radiation in the form of quanta is emitted or absorbed. This radiation energy equals the energy difference between the two orbits.

Problem 10

This has already been shown in equations (48), (49), and (50).

Problem 11

It has been found that the spectrum of the hydrogen atom is given as follows:

$$\overline{\nu} = R_y \left(\frac{1}{n_2^2} - \frac{1}{n_1^2} \right) \tag{51}$$

In (51), when $n_2 = 1, 2, 3, 4, 5, 6$, these are referred to as the Lyman, Paschen, Brackett, Pfund, and Humphreys series, respectively. It can be observed that as $n_1 \to \infty$, $\overline{\nu} \to R_y\, n_2^2$.

Problem 12

Light is absorbed by the medium through processes such as atomic excitation and photoionization, molecular photodissociation, and free-free absorption. In other words, a photon converts its energy into the kinetic energy or internal energy of the particle and is destroyed. On the other hand, the medium can emit light through processes like atomic de-excitation and radiative recombination, molecular de-excitation and creation, and free-free emission of free electrons. That is, a photon is generated from the kinetic energy or internal energy of the particles.

Problem 13

- **Radiative Excitation:** This occurs when an atom absorbs a photon. Absorption lines appear in the continuous spectrum. Typically, the excited state lasts for about 10^{-8} seconds, which is very short, and the atom soon re-emits the photon. The absorbed photon may be emitted as several lower-energy photons. Consequently, the input wavelength is transformed into a longer wavelength, and the spectrum corresponding to the input wavelength decreases. Furthermore, while the absorbed photons come from the direction of the light source, the emitted photons can go in any direction. As a result, photons of the wavelength that experienced absorption are observed to be fewer compared to others.
- **Collisional Excitation:** A free-moving particle (either an electron or an atom) collides with an atom and transfers part of its kinetic energy to the atom. If the energy transferred from one particle to another is equal to the transition energy of the electron, the atom becomes excited. The excited atom then emits a photon and returns to its ground state, creating an emission line spectrum.

Problem 14

- **Radiative De-excitation:** Atoms are always interacting with the electromagnetic field. An excited atom spontaneously relaxes to a lower energy level within approximately 10^{-8} seconds, during which a photon is emitted.

- **Collisional De-excitation:** This is the reverse process of collisional excitation. The colliding particle gains kinetic energy from the energy exchange.
- **Forbidden Transition:** A forbidden line occurs when an atom transitions from an excited metastable state to the ground state. Under most conditions, the atom will relax from the metastable state to the ground state through collisions before the radiative process can occur. Therefore, forbidden lines arise in low-density gases, where there are fewer opportunities for collisions during the time between excitation and radiative de-excitation.

Problem 15

According to (51), $E = 13.6\text{eV}$.

Problem 16

Pauli's exclusion principle states that no two identical fermions can occupy the same quantum state.

Problem 17

When an ion has at least one bound electron, its spectrum is produced similarly to that of a neutral atom. The spectrum of the ion closely resembles that of a neutral atom with the same number of outer electrons; however, due to the increased charge of the atomic nucleus, there is a shift in the overall wavelength.

Problem 18

Molecular transitions create numerous overlapping spectra. The transitions are as follows:

1. **Electronic Transition:** The electronic energy states of the bonded electron cloud surrounding the atomic nuclei.
2. **Vibrational Transition:** The distance between the atomic nuclei is quantized into discrete vibrational energy states. As a result, vibrational transitions occur; if the distance becomes too great, the bond between the atoms breaks, leading to the dissociation of the molecule.
3. **Rotational Transition:** Molecules rotate around multiple axes in space, resulting in discrete rotational energy states.

Problem 19

The mean kinetic energy of free particles can be derived using the Maxwell-Boltzmann distribution. It expressed the probability density function of the particle to move at a certain velocity. Let's derive starting from Maxwell-Boltzmann distribution.

For simplicity of the derivation, we will assume that Boltzmann's distribution of energy is true without proving it. It states that:

$$P(E) \propto e^{-E/k_B T} \tag{52}$$

It says that a probability that the energy particle is proportional to $\exp(-E/k_B T)$. Every probability density function undergoes normalization process. Let the probability density function be $f(x)$.

$$\int_0^\infty \int_0^\infty \int_0^\infty f(x) \mathrm{d}v_x \mathrm{d}v_y \mathrm{d}v_z = 1 \tag{53}$$

Let's use the coefficient for the probability function, k. In addition, infinitesimal volume of velocity can be converted into spherical coordinate so that we can easily manage the calculation. Here,

$$\mathrm{d}v_x \mathrm{d}v_y \mathrm{d}v_z = v^2 \sin\theta \mathrm{d}\theta \mathrm{d}\phi \mathrm{d}v \tag{54}$$

Therefore,

$$\int_0^\infty \int_0^\infty \int_0^\infty k e^{-mv^2/k_B T} v^2 \sin\theta \mathrm{d}\theta \mathrm{d}\phi \mathrm{d}v = 1 \tag{55}$$

After the calculation, we are left with:

$$k = \left(\frac{m}{2\pi k_B T}\right)^{3/2} \tag{56}$$

Therefore, the Maxwell-Boltzmann velocity distribution is given by:

$$f(v) = \left(\frac{m}{2\pi k_B T}\right)^{3/2} 4\pi v^2 \exp\left(-\frac{mv^2}{2k_B T}\right) \tag{57}$$

To find the root mean square velocity (v_{rms}), we use the formula:

$$v_{\mathrm{rms}}^2 = \int_0^\infty v^2 f(v) \, \mathrm{d}v \tag{58}$$

Substitute $f(v)$ into the integral:

$$v_{\mathrm{rms}}^2 = \int_0^\infty v^2 \left(\frac{m}{2\pi k_B T}\right)^{3/2} 4\pi v^2 \exp\left(-\frac{mv^2}{2k_B T}\right) \, \mathrm{d}v \tag{59}$$

Simplify the constants:

$$v_{\text{rms}}^2 = \left(\frac{m}{2\pi k_B T}\right)^{3/2} 4\pi \int_0^\infty v^4 \exp\left(-\frac{mv^2}{2k_B T}\right) \mathrm{d}v \tag{60}$$

Let $x = \frac{mv^2}{2k_B T}$, then $\mathrm{d}x = \frac{mv\,\mathrm{d}v}{k_B T}$. Also, $v^2 = \frac{2k_B T}{m}x$ and $v\,\mathrm{d}v = \frac{k_B T}{m}\,\mathrm{d}x$:

$$v_{\text{rms}}^2 = \left(\frac{m}{2\pi k_B T}\right)^{3/2} 4\pi \left(\frac{2k_B T}{m}\right)^2 \int_0^\infty x^2 \exp(-x)\,\mathrm{d}x \tag{61}$$

The integral $\int_0^\infty x^2 \exp(-x)\,\mathrm{d}x$ is a gamma function:

$$\int_0^\infty x^2 \exp(-x)\,\mathrm{d}x = 2! \tag{62}$$

Thus, the integral evaluates to 2, and we have:

$$v_{\text{rms}}^2 = \left(\frac{m}{2\pi k_B T}\right)^{3/2} 4\pi \left(\frac{2k_B T}{m}\right)^2 \cdot 2 \tag{63}$$

Simplify the expression:

$$v_{\text{rms}}^2 = \frac{3k_B T}{m} \tag{64}$$

Taking the square root to find v_{rms}:

$$v_{\text{rms}} = \sqrt{\frac{3k_B T}{m}} \tag{65}$$

The energy for the free particle is the kinetic energy. Therefore, we can express that kinetic energy using the root mean square velocity.

$$\therefore \frac{1}{2}mv_{\text{rms}}^2 = \frac{3}{2}k_B T \tag{66}$$

Problem 20

20.1

To derive the Boltzmann's equation that states the distribution of excited particles, we need to know what partition function (Z), and degeneracy factor (g) is.

First of all, degeneracy factor, g, is the contribution factor that is multiplied in front the probabilty. For example, let's say that we are rolling two dice. All possible outcome for the sum of the numbers are 2 to 12. But we do not say that the probability of the getting the sum to 7 is 1/36. We need to multiply all possible number of independent events that lead to it.

Secondly, the partition function, $Z(T)$, is defined as:

$$Z(T) \equiv \sum_i g_i e^{-E/k_B T} \tag{67}$$

The reason why this is defined is because it manages the total number of events. It is equivalent to 36 from the example above. The factor that is multiplied after the degeneracy factor is the Boltzmann distribution of the particle by the energy state.

Therefore the probability for the particle to have a energy state E_i is:

$$P(E_i) = \frac{g_i}{Z(T)} e^{-E_i/k_B T} \tag{68}$$

For two different energy states, Boltzmann's equation holds:

$$\frac{n_2}{n_1} = \frac{g_2}{g_1} e^{-(E_2 - E_1)/k_B T} \tag{69}$$

20.2

Let's assume various situations and see how Boltzmann's equation explains the result.

When temperature T increases, the denominator on the exponential increases, leading to increasing of the ratio n_2/n_1.

When the excitation energy between two level $(E_2 - E_1)$ decreases, the numerator of the exponential function decreases. This also leads to the increase of the ratio n_2/n_1.

Problem 21

Saha's equation explains the distribution of the ionized particle of jth and $j+1$th state. It is somehow different from the Boltzmann's equation that it explains ions instead of excited particles.
Assume a ground stated particle. The number density of the particle is n_0. And we will take a small portion of this particle from infintesimal volume and shoot energy.

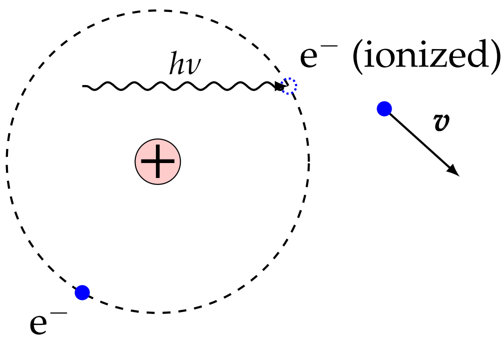


Figure 2: Ionization Process Due to External Energy Supply

Here, the energy difference is

$$\Delta E = h\nu = \chi + \frac{1}{2}m_e v^2 \tag{70}$$

where χ is the ionization energy of that particle.

$$\frac{\mathrm{d}n_+}{n_0} = \frac{g_+ g_e}{g_0} e^{-\Delta E/k_B T} \tag{71}$$

Here, g_e indicates a degeneracy factor of the electrons. And it is expressed as:

$$g_e = \frac{2\mathrm{d}V\mathrm{d}^3p}{h^3}. \tag{72}$$

Degeneracy factor of a electron explains the state of electron on the phase space. 2 multiplied indicates two different spins of electrons, $\mathrm{d}V$ indicates infinitesimal volume, d^3p indicates three different direction of the momentum, and h^3 is multiplied due to the Heisenberg's uncertainty principle.

Here, for the calculation,

$$\mathrm{d}V = \frac{1}{n_e} \tag{73}$$

where n_e is the number density of the electrons, and

$$\mathrm{d}^3p = \mathrm{d}p_x\mathrm{d}p_y\mathrm{d}p_z = 4\pi p^2\mathrm{d}p = 4\pi m_e^3 v^2\mathrm{d}v \tag{74}$$

Substitute every thing in, we get:

$$g_e = \frac{8\pi m_e^3 v^2}{n_e h^3}\mathrm{d}v \tag{75}$$

$$\frac{\mathrm{d}n_+}{n_0} = \frac{8\pi m_e^3 g_+}{n_e h^3 g_0} e^{-\Delta E/k_B T} v^2\mathrm{d}v \tag{76}$$

Since we are interested in the particle distribution regardless of their speed, we will integrate this from 0 to ∞. We will accept the fact that

$$\int_0^\infty x^2 e^{-x^2}\mathrm{d}x = \frac{\sqrt{\pi}}{4} \tag{77}$$

After all of those calculations, we are left with that:

$$\frac{n_+ n_e}{n_0} = \frac{2g_+}{g_0}\left(\frac{2\pi m_e k_B T}{h^2}\right)^{3/2} e^{-\chi/k_B T} \tag{78}$$

Notice that this is done for the ground state. To expand it into any excited state of the particle, we have to use the following relation.

$$\frac{n_{0,\text{ground}}}{n_0} = \frac{g_0}{Z(T)} \text{ and } \frac{n_{+,\text{ground}}}{n_+} = \frac{g_+}{Z_+(T)} \tag{79}$$

In addition to that, substituting any ionization state j and $j+1$ gives:

$$\frac{n_{j+1}n_e}{n_j} = \frac{2Z_{j+1}(T)}{Z_j(T)} \left(\frac{2\pi m_e k_B T}{h^2}\right)^{3/2} e^{-\chi_{j,j+1}/k_B T} \tag{80}$$

The better of utilizing this equation involves in introducing the electron pressure P_e. The reason to introduce this concept is because this can convert the formula into the function of temperature T. Due to ideal gas state equation,

$$P_e V = N k_B T \tag{81}$$

Therefore,

$$P_e = \frac{N}{V} k_B T = n_e k_B T \tag{82}$$

Let's conclude the derivation of the Saha's equation. The final form of the Saha's equation is:

$$\frac{n_{j+1}}{n_j} = \frac{2k_B T Z_{j+1}(T)}{P_e Z_j(T)} \left(\frac{2\pi m_e k_B T}{h^2}\right)^{3/2} e^{-\chi_{j,j+1}/k_B T} \tag{83}$$

21.1

Now, let's assume various situations to see how Saha's equation explains the result for each of the phenomenon.

When the temperature, T, increases, each of the multiplied terms increases. Thus it increases the ratio n_{j+1}/n_j.

When the electron density increases, the denominator of the right hand side increases, leading to the decrease of the ratio.

When the ionization energy increases, it increases the denominator of the exponential function. Therefore, it leads to the decreases of the ratio.

Problem 22

22.1

The reason to look at the ratio N_2/N_{tot} is because of the Balmer's line from the spectrum, which covers the visible ray from the entire spectrum.

N_{tot} can be expressed as

$$N_{\text{tot}} = N_I + N_{II} \tag{84}$$

Because hydrogen only consists of one electron. Therefore there are two energy states for it, H_I region and H_{II}. For N_{II}, we can approximate it as:

$$N_I \approx N_1 + N_2 \tag{85}$$

This is because slight excessive amount of energy can lead to full ionization of the electron instead of the excitement. Using all of those information, rewrite the equation, we get:

$$\frac{N_2}{N_{\text{tot}}} = \frac{N_2}{N_I}\frac{N_I}{N_I + N_{II}} = \left(\frac{N_2}{N_1 + N_2}\right)\left(\frac{N_I}{N_I + N_{II}}\right) \tag{86}$$

The left side of the multiplication, indicates a distribution of excited states, thus we use Boltzmann's distribution. And for the right side of the multiplication, it indicates a distribution of ions, thus we use Saha's equation for the analysis. Let's first convert it into a format to make the analysis easier.

$$\frac{N_2}{N_{\text{tot}}} = \left(\frac{N_2}{N_1 + N_2}\right)\left(\frac{N_I}{N_I + N_{II}}\right) = \left(\frac{N_2/N_1}{1 + N_2/N_1}\right)\left(\frac{1}{1 + N_{II}/N_I}\right) \tag{87}$$

22.2

For neutral hydrogen, the degeneracy is expressed as:

$$g_i = 2i^2 \tag{88}$$

In addition, we assume $Z_I \approx g_1 = 2$ and since H_{II} are free particles, the degeneracy factor is 1, i.e., $Z_{II} = 1$. Substitute everything we have derived so far, we are left with:

$$\frac{N_2}{N_{\text{tot}}} = \left(\frac{4\exp(-1.183 \times 10^5/T)}{1 + 4\exp(-1.183 \times 10^5/T)}\right)\left(\frac{P_e}{P_e + 0.332T^{5/2}\exp(-1.578 \times 10^5/T)}\right) \tag{89}$$

When we assume the electron pressure to be $P_e = 20\ \text{N/m}^2$, we get the following image of graph:

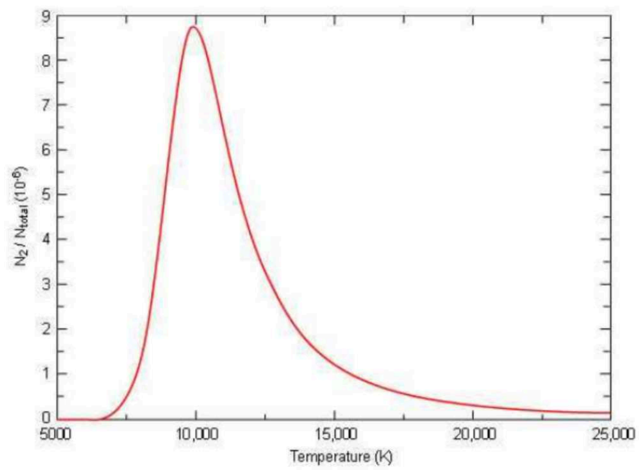


Figure 3: Ratio of Excited Hydrogen by Its Surface Temperature

According to the figure above, the ratio that we've found is maximized when the temperature is $T = 10,000$ K. This indicates that the hydrogen star with the temperature around $T = 10,000$ K will show maximum visible region spectrum.

22.3

When the temperature of the star decreases or increases from the temperature at $T = 10,000$ K, the intensity of the Balmer line will decreases, always.

Problem 23

The previous problem only coped with specific case, where it is a hydrogen star with specific electron pressure. If consisting elements of the star changes, or if the spectral type of the star changes, then the temperature that makes the intensity of the spectrum is not always $T = 10,000$ K. The graph for generalized stars is provided below:

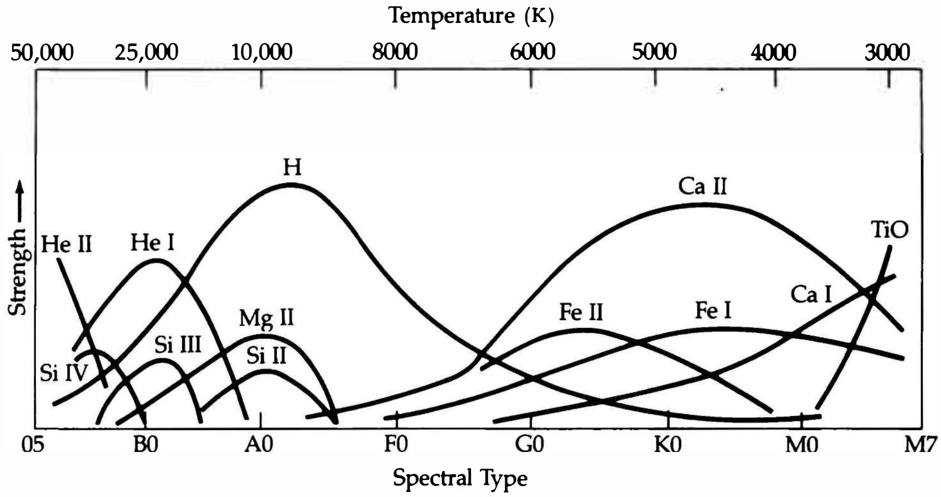


Figure 4: Various Elements' Spectral Intensity by Temperature

Problem 24

Natural Broadening is due to Heisenberg's unccertainty principle. Since the action 'observation' itself requires the energy, the uncertainty of energy of the particle and the uncertainty of the time cannot be 0 simultaneously. Heisenberg's uncertainty principle says:

$$\Delta E \Delta t \geq \frac{h}{4\pi} \tag{90}$$

Note that the spectral line broadening is due to the uncertainty (Δ) of the frequency. Therefore,

$$\Delta E = \frac{h}{4\pi \Delta t} \tag{91}$$

According to the quantum mechanics, $\Delta\lambda$ can be expressed as:

$$\begin{aligned}\Delta E &= \left|\frac{\mathrm{d}E}{\mathrm{d}\lambda}\right| \Delta\lambda \\ &= \frac{hc}{\lambda^2}\Delta\lambda\end{aligned}$$

Combining two former equations will provide:

$$\Delta\lambda = \frac{\lambda^2}{4\pi c\Delta t} \tag{92}$$

This formula well explains how on a visible region where the excited state remains for around 10^{-8} s has a spectral line broadening about 0.05 mÅ.

Problem 25

Thermal Doppler broadening is due to active particle's movement. Obviously, their movements gets more intense as the temperature increases. This relation can explain the thermal Doppler broadening. According to the equation of average kinetic energy:

$$\frac{1}{2}mv_{\mathrm{rms}}^2 = \frac{3}{2}k_B T \tag{93}$$

Here, if we measure the temperature, we are able to obtain the root square mean value of the particle's speed. This can be substituted into Doppler's equation:

$$\frac{\Delta\lambda}{\lambda} = \frac{v}{c} \tag{94}$$

Therefore:

$$\Delta\lambda = \frac{2}{c}\sqrt{\frac{3k_B T}{m}} \tag{95}$$

The reason why there is 2 multiplied in front of the equation is because the movement of the particles are random, causing the spectral line broadening of both sides.

Problem 26

When an external magnetic field is applied, the magnetic moment interacts with this field, causing an energy shift. This causes the spectral line to be separated into 3 or more lines. The energy shift is expressed as:

$$\Delta E = -\boldsymbol{\mu} \cdot \boldsymbol{B} = \frac{e}{2m_e}\boldsymbol{L} \cdot \boldsymbol{B} \tag{96}$$

And according to the quantum mechanics, this can be rewriten using the magnetic quantum number:

$$\Delta E = m_l \mu_B B \tag{97}$$

This results in the spectral separation of the lines, causing a line to be broadened out.